Problem set 2

The golden rule- and the optimal solution

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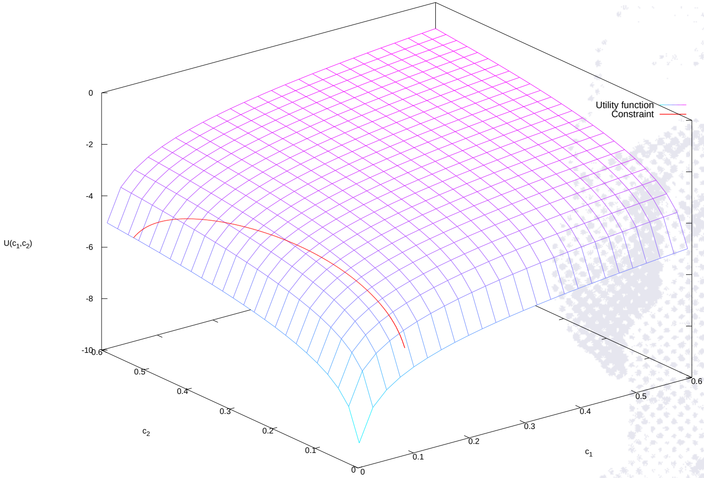
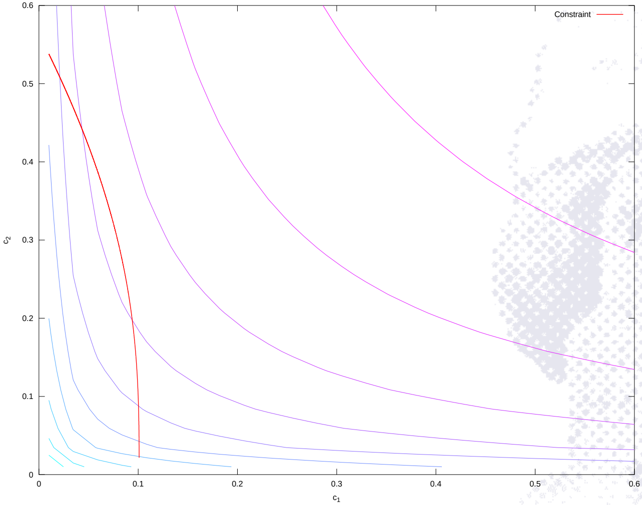
Contents

Contents

Two graphs corresponding to PS1



Two graphs corresponding to PS1



Contents

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Basic equations

- We start the model with the basic equations characterizing the economy.
- Those equations reflect the assumptions we impose on the model economy.
- The first assumption is that we deal with a closed economy, which means that there is no import or export and capital cannot be invested abroad.
- By this assumption the national income identity reduces to

$$Y_t = C_t + I_t,$$

  where Y_t denotes output, C_t denotes consumption and I_t denotes investment in period t.
- We denote the labor force in period t by N, hence to get per capita units we have to divide by N.

National income identity

- Dividing by N yields

$$\begin{aligned} \frac{Y_t}{N} &= \frac{C_t}{N} + \frac{I_t}{N} \\ y_t &= c_t + i_t. \end{aligned} \tag{1}$$

- Our notation is such that lower case letters denote variables in per capita units.
- Thus, y_t for example means output per capita.
- We will express the whole model in per capita units.

Capital accumulation

- A fundamental ingredient of the model is our assumption about the evolution of the capital stock K_t

$$\begin{aligned} K_{t+1} &= K_t + I_t - \delta K_t \\ \Leftrightarrow \Delta K_{t+1} \equiv K_{t+1} - K_t &= I_t - \delta K_t. \end{aligned}$$

- Please note that the population is assumed to be constant which makes the derivation easier at this point.
- We divide this equation by N as well, we get

$$\begin{aligned} \frac{\Delta K_{t+1}}{N} &= \frac{I_t}{N} - \delta \frac{K_t}{N} \\ \Delta k_{t+1} &= i_t - \delta k_t. \end{aligned} \tag{2}$$

Production function

- The remaining piece of our model is the production function.
- Different from the lecture we assume a specific functional form instead of using a general representation.
- The Cobb-Douglas production function is widely used among (macro)economists.
- We rewrite it in per capita units as well

$$
\begin{aligned}
Y_t = F(K_t, N) &= K_t^{\alpha} N^{1-\alpha} \\
\frac{Y_t}{N} = F\left(\frac{K_t}{N}, 1\right) &= \frac{K_t^{\alpha} N^{1-\alpha}}{N} \\
y_t = F(k_t) &= k_t^{\alpha}. \qquad (3)
\end{aligned}
$$

Please recall for the derivation that $N = N^{\alpha} N^{1-\alpha}$.

The dynamic constraint

- Now we are able to write down the dynamic constraint of the model.
- Therefore we use the results we have derived above.
- We combine the national income identity (1), the per capita capital accumulation equation (2) and the production function (3), this yields

$$\begin{aligned}
\Delta k_{t+1} &= i_t - \delta k_t \\
\Delta k_{t+1} &= y_t - c_t - \delta k_t \\
\Delta k_{t+1} &= k_t^{\alpha} - c_t - \delta k_t \\
\Leftrightarrow c_t &= k_t^{\alpha} - k_{t+1} + (1-\delta)k_t. \qquad (4)
\end{aligned}$$

- You can convince yourself easily that the equations we have derived here are particular cases of the general form derived in the lecture.

Properties of $F(\cdot)$ 1

- Next, we have a look at the properties of the Cobb-Douglas production function considered in this problem.
- Therefore we have to consider the following properties for $k > 0$

$$F(k) > 0, \quad F'(k) > 0, \quad F''(k) < 0$$

$$\lim_{k \to 0} F'(k) = \infty, \quad \lim_{k \to \infty} F'(k) = 0.$$

- The first one is trivial to see.
- It basically follows from the fact that the function is increasing for increasing $k > 0$.
- For the second property wee need the first derivative

$$F'(k) = \alpha k^{\alpha - 1}.$$

- For positive k the first derivative is always positive.

Properties of $F(\cdot)$ 2

- The second derivative is

$$F''(k) = (\alpha - 1)\alpha k^{\alpha-2}.$$

- Since $0 < \alpha < 1$ this must be negative.
- Now we let k approach zero in $F'(\cdot)$

$$\lim_{k\to 0} F'(k) = \lim_{k\to 0} \alpha k^{\alpha-1} = \lim_{k\to 0} \frac{\alpha}{k^{1-\alpha}} = \infty.$$

- Similarly we have

$$\lim_{k\to\infty} F'(k) = \lim_{k\to\infty} \alpha k^{\alpha-1} = \lim_{k\to\infty} \frac{\alpha}{k^{1-\alpha}} = 0.$$

Graphical illustration

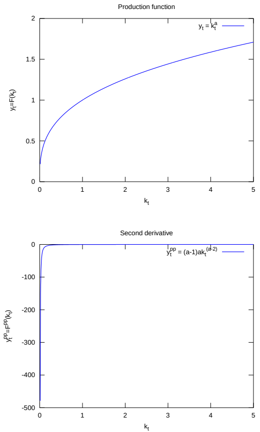

Figure: Cobb-Douglas production function

The golden rule solution 1

- As you already saw in the lecture, the golden rule solution is one where the discount factor equals unity.
- Hence, individuals value consumption in each period equal.
- Thus we must have the same level of consumption over time $c_t = c_{t+1} = c$.
- The dynamic constraint (4) becomes

$$
\begin{aligned}
c &= k^\alpha - k + (1-\delta)k \\
c &= k^\alpha - \delta k.
\end{aligned}
$$

- Now we differentiate with respect to k, we get

$$
\frac{\partial c}{\partial k} = \alpha k^{\alpha-1} - \delta \overset{!}{=} 0.
$$

The golden rule solution 2

- Since we have assumed a particular function for $F(\cdot)$ we are able to solve for the golden rule capital stock

$$\alpha k_{GR}^{\alpha-1} = \delta$$
$$k_{GR}^{\alpha-1} = \frac{\delta}{\alpha}$$
$$k_{GR} = \left(\frac{\delta}{\alpha}\right)^{\frac{1}{\alpha-1}} = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}.$$

- The golden rule level of consumption is then obtained by substituting into the golden rule constraint

$$c_{GR} = k_{GR}^{\alpha} - \delta k_{GR} = \left(\frac{\delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \delta\left(\frac{\delta}{\alpha}\right)^{\frac{1}{\alpha-1}}.$$

The golden rule solution 3

- We rewrite the solution for c_{GR} slightly

$$
\begin{aligned}
c_{GR} &= \left(\frac{\delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \delta\left(\frac{\delta}{\alpha}\right)^{\frac{1}{\alpha-1}} \\
&= \delta^{\frac{\alpha}{\alpha-1}} \cdot \left[\left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}\right] \\
&= \delta^{\frac{\alpha}{\alpha-1}} \cdot \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \cdot \left[1 - (\frac{1}{\alpha})^{-1}\right] \\
&= \delta^{\frac{\alpha}{\alpha-1}} \cdot \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \cdot (1-\alpha)
\end{aligned}
$$

- In the following we will analyze the solution by comparative statics.

Comparative statics of the golden rule solution 1

- For (i) We differentiate k_{GR} with respect to δ

$$
\begin{aligned}
\frac{\partial k_{GR}}{\partial \delta} &= \frac{1}{\alpha-1} \cdot \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} \cdot \delta^{\frac{1}{\alpha-1}-1} \\
&= \frac{1}{\alpha-1} \cdot \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} \cdot \delta^{\frac{2-\alpha}{\alpha-1}} \\
&= \underbrace{\frac{1}{\alpha-1}}_{<0} \cdot \underbrace{\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}}_{>0} \cdot \underbrace{\left(\frac{1}{\delta}\right)^{\frac{2-\alpha}{1-\alpha}}}_{>0} < 0
\end{aligned}
$$

Comparative statics of the golden rule solution 2

- Since
$$F''(k) = \alpha(\alpha - 1)k^{\alpha-2}$$
we get
$$\begin{aligned}
F''(k_{GR}) &= \alpha(\alpha-1)\left[\left(\frac{\delta}{\alpha}\right)^{\frac{1}{\alpha-1}}\right]^{\alpha-2} \\
&= \alpha(\alpha-1)\left(\frac{\delta}{\alpha}\right)^{\frac{\alpha-2}{\alpha-1}} \\
&= \alpha(\alpha-1)\left(\frac{\delta}{\alpha}\right)^{\frac{2-\alpha}{1-\alpha}} \\
&= \alpha^{-\frac{1}{1-\alpha}}(\alpha-1)\delta^{\frac{2-\alpha}{1-\alpha}} \\
&= \alpha^{\frac{1}{\alpha-1}} \cdot (\alpha-1) \cdot \delta^{\frac{2-\alpha}{1-\alpha}}.
\end{aligned}$$

Comparative statics of the golden rule solution 3

- Hence,

$$\frac{\partial k_{GR}}{\partial \delta} = \frac{1}{F''(k_{GR})}.$$

- For the second part (ii) we have

$$\begin{aligned}
\frac{\partial c_{GR}}{\partial \delta} &= \frac{\alpha}{\alpha - 1} \cdot \delta^{\frac{\alpha}{\alpha-1}-1} \cdot \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \cdot (1-\alpha) \\
&= \delta^{\frac{1}{\alpha-1}} \cdot \frac{\alpha}{\alpha-1} \cdot \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \cdot (1-\alpha) \\
&= -\delta^{\frac{1}{\alpha-1}} \cdot \alpha \cdot \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \\
&= \ldots
\end{aligned}$$

Comparative statics of the golden rule solution 4

$$\ldots = -\delta^{\frac{1}{\alpha-1}} \cdot \left(\frac{1}{\alpha}\right)^{-1} \cdot \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}$$
$$= -\delta^{\frac{1}{\alpha-1}} \cdot \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} < 0.$$

- Hence,

$$\frac{\partial c_{GR}}{\partial \delta} = -k_{GR}.$$

Contents

The problem

- Objective

$$\max_{\{c_{t+s},k_{t+s+1}\}_{s=0}^{\infty}} V_t = \max_{\{c_{t+s},k_{t+s+1}\}_{s=0}^{\infty}} \sum_{s=0}^{\infty} \beta^s \ln c_{t+s} \tag{5}$$

subject to

$$c_t + k_{t+1} = k_t^{\alpha} + (1-\delta)k_t \quad \text{for all } t \geq 0. \tag{6}$$

- Again we use the Lagrangian to solve the problem.
- The only difference is that we have an *infinite* number of first-order conditions.
- The Lagrangian to this problem is

$$\mathcal{L} = \sum_{s=0}^{\infty} \beta^s \ln c_{t+s} + \lambda_{t+s} \left[k_{t+s}^{\alpha} + (1-\delta)k_{t+s} - c_{t+s} - k_{t+s+1}\right]. \tag{7}$$

The Lagrangian

- We have to take partial derivatives with respect to c_{t+s} and k_{t+s+1}.
- We write the sum as

$$
\begin{aligned}
\mathcal{L} = {} & \beta^0 \ln c_t + \lambda_t \left[k_t^\alpha + (1-\delta)k_t - c_t - k_{t+1}\right] \\
& + \beta^1 \ln c_{t+1} + \lambda_{t+1} \left[k_{t+1}^\alpha + (1-\delta)k_{t+1} - c_{t+1} - k_{t+2}\right] + \cdots \\
\cdots & + \beta^s \ln c_{t+s} + \lambda_{t+s} \left[k_{t+s}^\alpha + (1-\delta)k_{t+s} - c_{t+s} - k_{t+s+1}\right] \\
& + \beta^{s+1} \ln c_{t+s+1} + \\
& + \lambda_{t+s+1} \left[k_{t+s+1}^\alpha + (1-\delta)k_{t+s+1} - c_{t+s+1} - k_{t+s+2}\right] + \cdots .
\end{aligned}
$$

- Now we can see where we find c_{t+s} and k_{t+s+1}.
- c_{t+s} appears only in the third line.
- k_{t+s+1} appears in the third and fourth line.
- For a moment we can forget the following lines.

FOC

- The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_{t+s}} = \beta^s \frac{1}{c_{t+s}} - \lambda_{t+s} \overset{!}{=} 0 \tag{I}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+s+1}} = \lambda_{t+s+1}\left[\alpha k_{t+s+1}^{\alpha-1} + 1 - \delta\right] - \lambda_{t+s} \overset{!}{=} 0, \tag{II}$$

$\forall s = 0, 1, 2, \ldots, \infty.$

- Note that actually we have infinitely many FOCs.
- Rewriting (II) yields

$$\lambda_{t+s} = \lambda_{t+s+1}\left(\alpha k_{t+s+1}^{\alpha-1} + 1 - \delta\right). \tag{8}$$

- We can forward (I) in order to get

$$\lambda_{t+s} = \beta^s \frac{1}{c_{t+s}} \Leftrightarrow \lambda_{t+s+1} = \beta^{s+1} \frac{1}{c_{t+s+1}}. \tag{9}$$

Euler equation

- Substituting (9) into (8) gives

$$\frac{1}{c_{t+s}} = \left(1 + \alpha k_{t+s+1}^{\alpha-1} - \delta\right) \beta \frac{1}{c_{t+s+1}}. \tag{10}$$

- Equation (10) is the Euler equation for the infinite horizon case.
- The interpretation is analogously to the two period case.
- The Euler equation describes the optimal consumption path.
- It equates the marginal consumption today with the marginal consumption tomorrow (from saving) discounted by β.
- In the optimum the consumer cannot improve her utility by shifting consumption intertemporally.

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