Problem set 1

Consumption, saving, and the business cycle

Markus Roth

Chair for Macroeconomics
Johannes Gutenberg Universität Mainz

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The consumption function

- Consider the consumption function

$$C = c_0 + c_1 Y^d. \tag{1}$$

- This function should be familiar to you from undergraduate courses in macroeconomics.
- C denotes the aggregate consumption level in the economy.
- It is sometimes called *"Keynesian consumption function"*.
- Y^d is disposable income in the economy.
- c_0 and c_1 are parameters characterizing the function.
- Usually we restrict the parameters such that
  - $c_0 > 0$
  - $0 < c_1 < 1$.

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Discussing the consumption function

- c_0 is called *autonomous consumption*.
- I.e. consumption if disposable income is zero.
- We can have positive consumption in the presence of zero disposable income because of dissaving.
- Disposable income is defined as income net of taxes, i.e. $Y^d \equiv Y - T$.
- For consumption only the disposable income is relevant because only this part of income can actually be consumed.
- c_1 is called the *marginal propensity to consume* (MPC).
- It is a natural assumption that $0 < c_1 < 1$ because this means consumption is a fraction of income.
- We cannot consume more than we earn.
- On the other hand, higher income should yield higher consumption.

Problems with the function

- The Keynesian consumption function is the traditional way economists think of consumption.
- However, there are some problems to this function.
- The most pronounced problem is that it links consumption to current income and disregards potential future earnings.
- Lifetime income should be relevant for individuals' consumption saving decision.
- Another point is that the function per se is not micro-founded, it is set up by (reasonable) assumptions and empirical support.
- By contrast, modern macroeconomic theory is based on optimization problems of households, firms, central banks, ...
- In the tutorials we will derive a micro-based justification of the consumption function, where we substitute current income by permanent (lifetime) income.

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Recall for problem 2 and 3 from the lecture

- The capital accumulation equation is given by

$$k_{s+1} = k_s + i_s - \delta k_s.$$

- The national accounting identity (for a closed economy) is

$$y_s = c_s + i_s.$$

- Putting both together yields

$$\begin{aligned} y_s &= c_s + k_{s+1} - (1-\delta)k_s \\ \Leftrightarrow c_s + k_{s+1} &= y_s + (1-\delta)k_s. \end{aligned} \tag{2}$$

- The consumer can shift resources intertemporally by investing in the capital stock.

The objective function

- The objective function reads

$$\max_{c_1,c_2,k_2} V_1 = \max_{c_1,c_2,k_2} \log c_1 + \beta \log c_2. \tag{3}$$

- This means that the household chooses consumption in period 1 (c_1) and period 2 (c_2) in order to maximize its lifetime utility function which we call V_1.
- In this problem the lifetime of the household is two periods only.
- One can think of two periods where the household is young and old respectively.
- $0 < \beta < 1$ is the discount factor.
- It represents impatience of the household.

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The objective function/optimization problem

- The utility the household gets from consuming in period 1 or 2 is determined by the period utility function $U(\cdot)$.
- In our particular case the objective function is logarithmic.
- We usually assume that the objective function is concave.
- Why we make this assumption will be discussed in later tutorials.
- The household now chooses in every period of life how much to consume and how much to save for the next period.
- There is a trade-off between consumption today (period 1) and tomorrow (period 2).
- The household could consume more (save less) today but then it has to consume less tomorrow (because of the low savings).
- The trade-off comes by the budget constraints.

The constraints

- We maximize the lifetime utility function V_1 with respect to the budget constraints of the household.
- If there would not be any constraint the household could simply maximize its lifetime utility by consuming an infinite amount of c_1 and c_2.
- However, this is not reasonable hence we maximize subject to

$$c_1 + k_2 = k_1^{\alpha} + (1-\delta)k_1 \tag{4}$$
$$c_2 = k_2^{\alpha} + (1-\delta)k_2. \tag{5}$$

Solution of the problem

- In general there are three methods to solve such a dynamic optimization problem.
  1. Substitute the constraints into the objective function and compute the first derivative with respect to k_2.
  2. Set up the Lagrangian function $\mathcal{L}$ and compute the first derivative with respect to c_1, c_2 and k_2.
  3. Use the *Bellman equation* to solve the problem (in this case trivial).
- We will usually use the Lagrangian to solve the problem.

$$\begin{aligned} \mathcal{L} = \log c_1 + \beta \log c_2 + \lambda_1 \left[k_1^\alpha + (1-\delta)k_1 - c_1 - k_2\right] \\ + \lambda_2 \left[k_2^\alpha + (1-\delta)k_2 - c_2\right] \end{aligned} \tag{6}$$

- Note, that there are more than one possibility to set up the Lagrangian. We will see different approaches throughout the tutorials.
- However, the solution will always be identical, only the interpretation of the Lagrange multipliers (λ_1, λ_2) changes.

FOC

- The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} - \lambda_1 \overset{!}{=} 0 \tag{I}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \beta \frac{1}{c_2} - \lambda_2 \overset{!}{=} 0 \tag{II}$$

$$\frac{\partial \mathcal{L}}{\partial k_2} = -\lambda_1 + \lambda_2 \left[\alpha k^{\alpha-1} + (1-\delta) \right] \overset{!}{=} 0. \tag{III}$$

- Rearranging (III) yields

$$\lambda_1 = \lambda_2 \left[\alpha k^{\alpha-1} + 1 - \delta \right].$$

- Substituing (I) and (II) gives

$$\frac{1}{c_1} = \beta \frac{1}{c_2} \left[1 + \alpha k_2^{\alpha-1} - \delta \right] \tag{7}$$

Euler equation

- Equation (7) is known as the Euler equation.
- The Euler equation describes the optimal consumption path.
- It equates the marginal consumption today with the marginal consumption tomorrow (from saving) discounted by β.
- Note that $1 + \alpha k_{t+1}^{\alpha-1} - \delta$ can be interpreted as an interest rate.
- In the optimum the consumer cannot improve her utility by shifting consumption intertemporally.
- We can rewrite the equation to

$$\frac{U'(c_1)}{\beta U'(c_2)} = 1 + \alpha k_{t+1}^{\alpha-1} - \delta. \tag{8}$$

- Here we equate the marginal rate of substitution between consumption today and tomorrow (LHS) and the marginal rate of transformation (RHS).

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Generalizing the problem

- We now turn to a problem where the lifetime of households has infinitely many periods.
- This is more general than the model in the previous question.
- We can rationalize the infinite horizon setup with household bequests to younger generations.
- Hence, we view the representative household as a "family dynasty".
- However, the basic solution strategy and the economic implications do not change.

The problem

- Objective

$$\max_{\{c_{t+s},k_{t+s+1}\}_{s=0}^{\infty}} V_t = \max_{\{c_{t+s},k_{t+s+1}\}_{s=0}^{\infty}} \sum_{s=0}^{\infty} \beta^s \log c_{t+s} \tag{9}$$

subject to

$$c_s + k_{s+1} = k_s^{\alpha} + (1-\delta)k_s \quad \text{for all } s \geq 0. \tag{10}$$

- Again we use the Lagrangian to solve the problem.
- The only difference is that we have an *infinite* number of first-order conditions.
- The Lagrangian to this problem is

$$\mathcal{L} = \sum_{s=0}^{\infty} \beta^s \left\{\log c_{t+s} + \lambda_{t+s}\left[k_{t+s}^{\alpha} + (1-\delta)k_{t+s} - c_{t+s} - k_{t+s+1}\right]\right\}. \tag{11}$$

The Lagrangian

- We have to take partial derivatives with respect to c_{t+s} and k_{t+s+1}.
- We write the sum as

$$
\begin{aligned}
\mathcal{L} = \beta^0 \{\log c_t + \lambda_t [k_t^\alpha + (1-\delta)k_t - c_t - k_{t+1}]\} \\
+ \beta^1 \{\log c_{t+1} + \lambda_{t+1} [k_{t+1}^\alpha + (1-\delta)k_{t+1} - c_{t+1} - k_{t+2}]\} + \cdots \\
\cdots + \beta^s \{\log c_{t+s} + \lambda_{t+s} [k_{t+s}^\alpha + (1-\delta)k_{t+s} - c_{t+s} - k_{t+s+1}]\} \\
+ \beta^{s+1} \{\log c_{t+s+1} + \\
+ \lambda_{t+s+1} [k_{t+s+1}^\alpha + (1-\delta)k_{t+s+1} - c_{t+s+1} - k_{t+s+2}]\} + \cdots .
\end{aligned}
$$

- Now we can see where we find c_{t+s} and k_{t+s+1}.
- c_{t+s} appears only in the third line.
- k_{t+s+1} appears in the third and fourth line.
- For a moment we can forget the following lines.

FOC

- The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_{t+s}} = \beta^s \left(\frac{1}{c_{t+s}} - \lambda_{t+s} \right) \stackrel{!}{=} 0 \tag{I}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+s+1}} = \beta^{s+1} \lambda_{t+s+1} \left[\alpha k_{t+s+1}^{\alpha-1} + 1 - \delta \right] - \beta^s \lambda_{t+s} \stackrel{!}{=} 0, \tag{II}$$

$\forall s = 0, 1, 2, \dots, \infty$.

- Note that actually we have infinitely many FOCs.
- Rewriting (II) yields

$$\lambda_{t+s} = \beta \lambda_{t+s+1} \left(\alpha k_{t+s+1}^{\alpha-1} + 1 - \delta \right). \tag{12}$$

- We can forward (I) in order to get

$$\lambda_{t+s} = \frac{1}{c_{t+s}} \Leftrightarrow \lambda_{t+s+1} = \frac{1}{c_{t+s+1}}. \tag{13}$$

Euler equation

- Substituting (13) into (12) gives

$$\frac{1}{c_{t+s}} = \left(1 + \alpha k_{t+s+1}^{\alpha-1} - \delta\right) \beta \frac{1}{c_{t+s+1}}. \qquad (14)$$

- Equation (14) is the Euler equation for the infinite horizon case.
- The interpretation is analogously to the two period case.
- The Euler equation describes the optimal consumption path.
- It equates the marginal consumption today with the marginal consumption tomorrow (from saving) discounted by β.
- In the optimum the consumer cannot improve her utility by shifting consumption intertemporally.
- Again we could rewrite it to equate the MRS and MRT.

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Important properties of functions

- For this kind of analysis and also for other considerations we need to know some basic properties about utility- and production functions.
- A utility function $U(\cdot)$ usually is assumed to be concave.
- $U'(\cdot) > 0$ means that "more is always better".
- $U''(\cdot) < 0$ means that an additional unit of the argument (e.g. consumption) increases utility but to a smaller extent than the unit before.
- Marginal utility is positive but diminishing in the argument.
- We impose a similar assumption on the production function.
- We assume that $F(k_t)$ is also concave.
- Hence, an increase in k_t increases production $F(\cdot)$ but decreases marginal production $F'(\cdot)$.

Some examples

- If you have problems with the general definition of concavity consider the following examples:
  - Log: $U(c_t) = \ln(c_t)$
  - Power: $U(c_t) = \frac{c_t^{1-\sigma}-1}{1-\sigma}$ (with $\sigma > 0$)
  - Exponential: $-\mathrm{e}^{-\rho c_t}$ (with $\rho > 0$)
  - Production: $F(k_t) = k^{\alpha}$ (with $0 < \alpha < 1$)
- Try to plot those functions with a program or by using pen and paper to get intuition for the shape.

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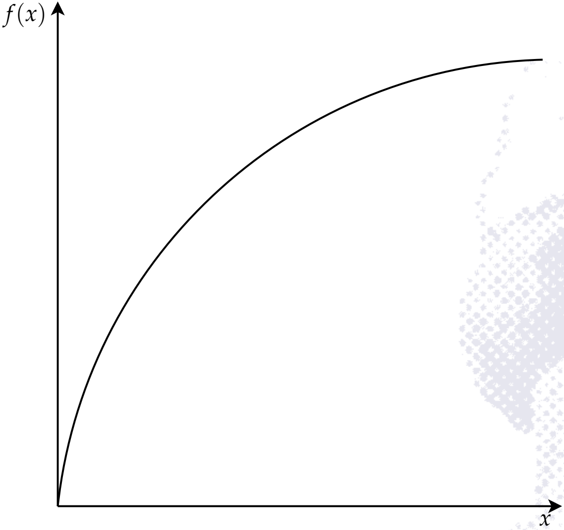

Figure: Concave function

Approximating the Euler equation

- Consider the expression of the optimality conditon (Euler equation) you have derived in the lecture

$$\frac{\beta U'(c_{t+1})}{U'(c_t)}\left[F'(k_{t+1}) + 1 - \delta\right] = 1. \tag{15}$$

- Furthermore consider the capital accumulation equation

$$\Delta k_{t+1} \equiv k_{t+1} - k_t = F(k_t) - \delta k_t - c_t. \tag{16}$$

- Note that you have seen both equations in the lecture.
- We take a first order Taylor approximation of $U'(c_{t+1})$ around c_t

$$U'(c_{t+1}) \simeq U'(c_t) + \Delta c_{t+1} U''(c_t).$$

- Such an approximation is "good" in the neighborhood of c_t.

Rearranging

- Dividing by $U'(c_t)$ yields

$$\frac{U'(c_{t+1})}{U'(c_t)} \simeq 1 + \frac{U''(c_t)}{U'(c_t)} \Delta c_{t+1}. \tag{17}$$

- We know that by (reasonable) assumption

$$\frac{U''(c_t)}{U'(c_t)} \leq 0.$$

- We rearrange (15)

$$\frac{U'(c_{t+1})}{U'(c_t)} = \frac{1}{\beta \left[F'(k_{t+1}) + 1 - \delta \right]}. \tag{18}$$

Substituting

- We substitute (17) for the left hand side in (18) and get

$$\Delta c_{t+1} = -\frac{U'(c_t)}{U''(c_t)} \left[1 - \frac{1}{\beta \left[F'(k_{t+1}) + 1 - \delta \right]} \right]. \tag{19}$$

  (typo in [Wickens, 2008]!)

- The capital accumulation equation was given by

$$\Delta k_{t+1} = F(k_t) - \delta k_t - c_t. \tag{16}$$

- With equations (19) and (16) we have a two-variable system of two (still nonlinear) difference equations.
- Since the system consists of two nonlinear difference equations there is no easy way to solve them analytically.
- However, we can use *phase diagrams* to understand the system.

Zero motion line 1

- First we determine the loci where $\Delta k_{t+1} = 0$ and $\Delta c_{t+1} = 0$.
- We call the result *zero motion lines*.

$$0 = -\frac{U'(c_t)}{U''(c_t)} \left[1 - \frac{1}{\beta \left[F'(k_{t+1}) + 1 - \delta\right]}\right]$$

$$\Leftrightarrow F'(k_{t+1}) = \underbrace{\frac{1-\beta}{\beta}}_{\equiv \theta} + \delta$$

$$F'(k_{t+1}) = \theta + \delta$$

- This equation implicitly defines a constant zero motion line where $\Delta c_{t+1} = 0$, i.e. consumption does not change.

Digression: discounting the future

- There are two different ways to express that agents are impatient.
- Usually we assume a discount factor $0 < \beta < 1$.
- Thus future period utility functions are multiplied by this factor expressing that utility tomorrow is worth less than utility today.
- However, sometimes it is convenient to think of impatience as "discounting" the future.
- Thus, we need a concept similar to the concept of interest rates, where future values are dicounted by $(1+\theta)^{-1}$.
- We can write

$$\beta = \frac{1}{1+\theta} \Leftrightarrow \theta = \frac{1-\beta}{\beta}.$$

Zero motion line 1

- Recall that on the zero motion line $\Delta c_{t+1} = 0$.

$$\Delta c_{t+1} = -\frac{U'(c_t)}{U''(c_t)} \left[1 - \frac{1}{\beta \left[F'(k_{t+1}) + 1 - \delta\right]}\right]. \tag{19}$$

- In equation (19) suppose starting from the zero motion line we increase k_t a little bit.
- What sign does Δc_{t+1} then have?
  $\Rightarrow$ It is negative.
- Why is this?
  $\Rightarrow$ If k_{t+1} increases $F'(k_{t+1})$ decreases ($F'(\cdot)$ is concave).
  $\Rightarrow$ Then the fraction (without minus sign) increases.
  $\Rightarrow$ Since U'/U'' is negative, the whole expression decreases.
  $\Rightarrow$ $\Delta c_{t+1} < 0$ when we increase k_{t+1}.
- The opposite is true when we decrease k_{t+1}.
- We bring this information into a k_{t+1}-c_t-diagram.

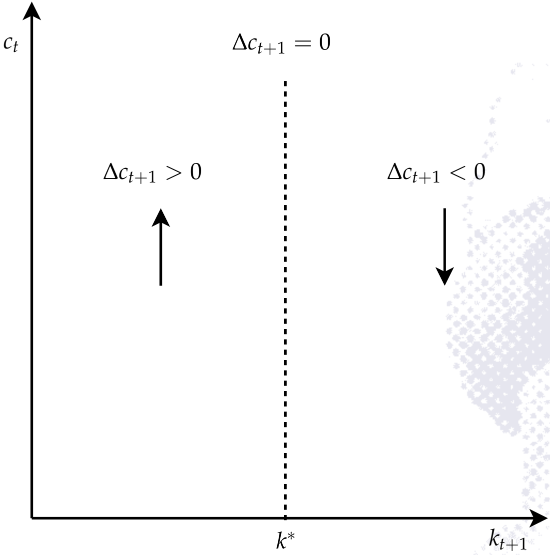

Figure: Δc_{t+1} diagram

Zero motion line 2

- The second zero motion line is found by setting $\Delta k_{t+1} = 0$

$$
\begin{aligned}
0 &= F(k_t) - \delta k_t - c_t \\
c_t &= F(k_t) - \delta k_t.
\end{aligned}
$$

- This is a concave function.
- Consider from $\Delta k_{t+1} = 0$ an increase in c_t in equation (16)

$$\Delta k_{t+1} = F(k_t) - \delta k_t - c_t. \tag{16}$$

- We find that $\Delta k_{t+1} < 0$.
- We bring this two another diagram in the same k_{t+1}-c_t-space.

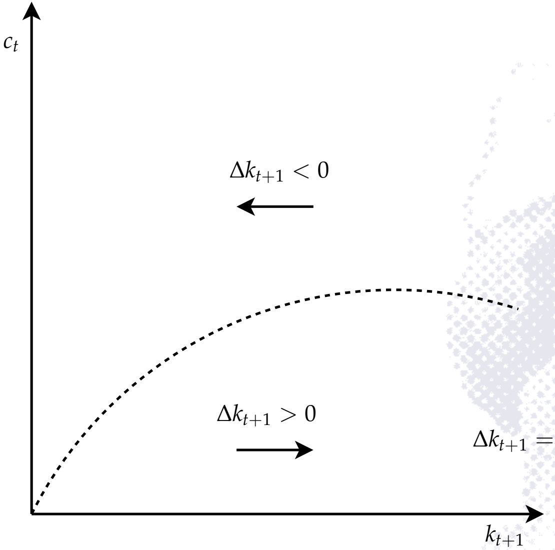

Figure: Δk_{t+1} diagram

Combining both diagrams

- We combine both diagrams.
- Therefore we use all information we have accumulated by our analysis.
- What we can see from the resulting diagram is that there is an intersection point where $\Delta k_{t+1} = \Delta c_{t+1} = 0$.
- We usually call this point the *steady state*.
- Furthermore we can draw a stable arm which has the property that the system moves towards the steady state.
- If we are not on this line, the system does not converge to the steady state.

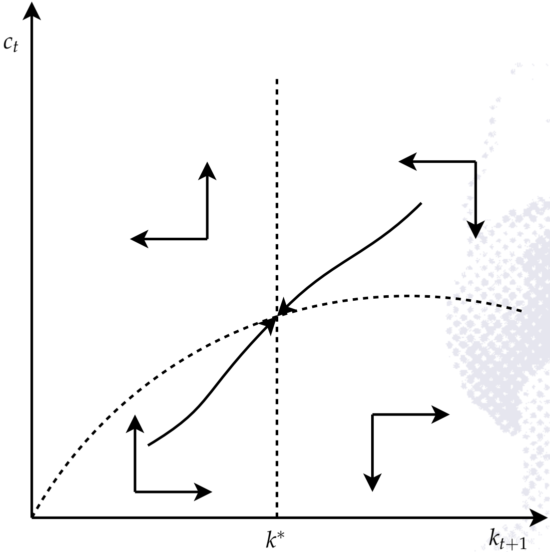

Figure: Combined diagram

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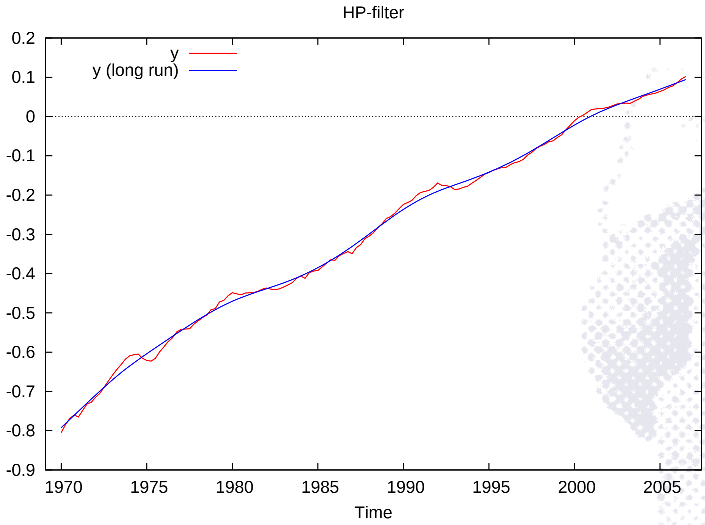
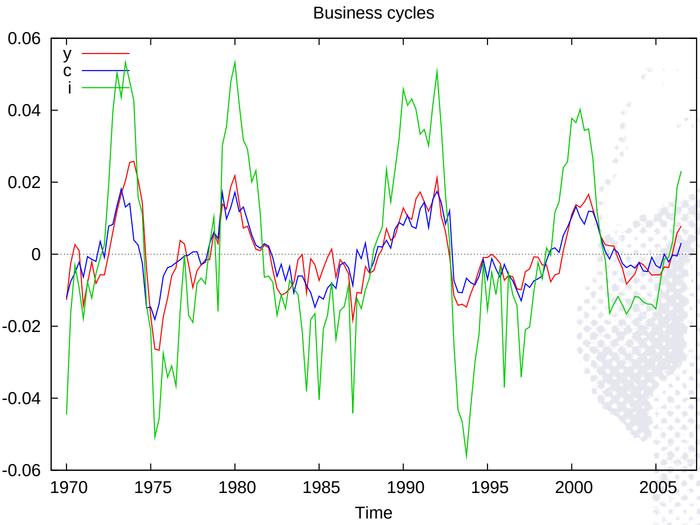
Agenda

- Open the dataset and inspect the content.
- Take logs of variables.
- Detrend the data using the HP filter.
- Compute the long-run and short-run component.
- Plot both components.
- Replicate the *stylized facts* of business cycles.
  - $\Rightarrow$ Compute standard deviations and correlation coefficients.


HP-filter
y
y (long run)
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
1970
1975
1980
1985
1990
1995
2000
2005
Time



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Some stylized facts of business cycles

1. Investment is much more volatile than output.
2. (Private) consumption and investment are strongly correlated with output.
   - ⇒ There is co-movement between the three variables.
3. There is persistence in output and consumption.

Contents

The minimization problem

- The minimization problem of the HP filter is

$$\min_{\{y_t^{lr}\}} \sum_{t=1}^{T} \left(y_t - y_t^{lr}\right)^2 + \lambda \sum_{t=2}^{T-1} \left[\left(y_{t+1}^{lr} - y_t^{lr}\right) - \left(y_t^{lr} - y_{t-1}^{lr}\right)\right]^2. \quad (20)$$

- The HP filter is a compromise between the two objectives.
  - Minimize the square deviation of the short-run component to trend.
  - Minimize the square change in the growth rate of the long-run component
- We are free to choose the relative weight λ of both objectives.
- For illustration we consider the following two extreme cases
  - $\lambda = 0$
  - $\lambda \to \infty$.

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$\lambda = 0$

- For $\lambda = 0$ the second objective is “switched off”.
- This means that we are only interested in minimizing the squared deviation between y_t and its long-run component y_t^{lr}.
- Since we choose y_t^{lr} in order to achieve our objective we set $y^{lr} = y_t$.
- This means in turn that we interpret the actual time series y_t as consisting solely of a long-run component.
- At the same time we decide that there is no short-run component in the actual time series y_t.

$\lambda \to \infty$

- For $\lambda \to \infty$ the first objective is “switched off”.
- This means that we are only interested in minimizing the squared change in y_t^{lr}.
- In the limiting case this means that we assume a constant change in y_t^{lr}.
- Thus also the growth rate of y_t^{lr} is assumed to be constant.
- The long-run component follows a linear time trend.

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Conclusion

- Of course, we choose some value between both extreme cases.
- Hence, we find the optimal compromise between both objectives.
- There is no “right” choice of λ but most researches agree with $\lambda = 1600$ for quartely data.

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Solving the minimization problem

- In order to solve the problem we have to distinguish five different cases.
- This means that the derivatives for some periods are different.
- More precisely, we compute the derivatives with respect to y_1^{lr}, y_2^{lr}, y_t^{lr}, y_{T-1}^{lr} and y_{T-2}^{lr}.
- The first order condition with respect to y_1^{lr} is given by

$$-2(y_1 - y_1^{lr}) + 2\lambda(y_3^{lr} - 2y_2^{lr} + y_1^{lr}) \stackrel{!}{=} 0.$$

- We solve this expression for y_1, this yields

$$y_1 = y_1^{lr}(1+\lambda) - 2\lambda y_2^{lr} + \lambda y_3^{lr}. \tag{21}$$

- We do the same for the remaining first order conditions.

The remaining periods

- The derivative with respect to y_2^{lr} is given by

$$-2(y_2 - y_2^{lr}) + 2\lambda(y_3^{lr} - 2y_2^{lr} + y_1^{lr})(-2) + 2\lambda[y_4^{lr} - 2y_3^{lr} + y_2^{lr}] \stackrel{!}{=} 0.$$

- Solving this for y_2 gives

$$y_2 = -2\lambda y_1^{lr} + (1 + 5\lambda)y_2^{lr} - 4\lambda y_3^{lr} + \lambda y_4^{lr}. \tag{22}$$

- The derivative with respect to y_t^{lr} reads

$$-2(y_t - y_t^{lr}) + 2\lambda(y_t^{lr} - 2y_{t-1}^{lr} + y_{t-2}^{lr}) + \cdots$$
$$\cdots 2\lambda(y_{t+1}^{lr} - 2y_t^{lr} + y_{t-1}^{lr})(-2) + 2\lambda(y_{t+2}^{lr} - 2y_{t+1}^{lr} + y_t^{lr}) \stackrel{!}{=} 0.$$

- We solve again for y_t

$$y_t = \lambda y_{t-2}^{lr} - 4\lambda y_{t-1}^{lr} + (1 + 6\lambda)y_t^{lr} - 4\lambda y_{t+1}^{lr} + \lambda y_{t+2}^{lr}. \tag{23}$$

The remaining periods

- We do not derive the remaining two derivatives (since the problem is symmetric), they are given by

$$y_{T-1} = \lambda y^{lr}_{T-3} - 4\lambda y^{lr}_{T-2} + (1+5\lambda)y^{lr}_{T-1} - 2\lambda y^{lr}_{T} \tag{24}$$
$$y_{T} = \lambda y^{lr}_{T-2} - 2\lambda y_{T-1} + (1+\lambda)y^{lr}_{T}. \tag{25}$$

- Having computed the first order conditions, we can state them in matrix notation.
- We define the following $T \times 1$ (column) vectors

$$\boldsymbol{y} \equiv \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix} \quad \text{and} \quad \boldsymbol{y}^{lr} \equiv \begin{pmatrix} y^{lr}_1 \\ y^{lr}_2 \\ \vdots \\ y^{lr}_T \end{pmatrix}. \tag{26}$$

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Matrix notation

- In addition we define the $T \times T$ matrix

$$\boldsymbol{A} \equiv$$

$$\begin{pmatrix} 1+\lambda & -2\lambda & \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2\lambda & 1+5\lambda & -4\lambda & \lambda & 0 & \cdots & 0 & 0 & 0 \\ \lambda & -4\lambda & 1+6\lambda & -4\lambda & \lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4\lambda & 1+5\lambda & -2\lambda \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda & -2\lambda & 1+\lambda \end{pmatrix}.$$

- The first two rows of this matrix contain the derivatives with respect to y_1^{lr} and y_2^{lr}, the last two rows contain the derivative with respect to y_{T-1}^{lr} and y_{T-2}^{lr} and the remaining $T-4$ rows contain the derivative with respect to y_t^{lr} on the "diagonal band".
- We can write the system as

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{y}^{lr}.$$

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Matrix notation

- Solving the system for $\boldsymbol{y}^{lr}$ yields
$$\boldsymbol{y}^{lr} = \boldsymbol{A}^{-1}\boldsymbol{y}.$$
we have derived a closed form solution for the HP-filter.
- Recall that $\boldsymbol{y}$ is a given data vector and $\boldsymbol{A}^{-1}$ only depends on λ which we are free to choose (recall the previous discussion).
- We are now prepared to implement the HP-filter into a matrix/vector based programming language.
- However, most statistical software packages already contain the HP-filter.
- If we are interested in the time series of the cyclical component $\{y_t^{sr}\}_{t=1}^T$ we simply use the "residual"

$$\begin{aligned} y_t &= y_t^{lr} + y_t^{sr} \\ \Leftrightarrow y_t^{sr} &= y_t - y_t^{lr}. \end{aligned} \tag{27}$$

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Interpretation of the problem

- Similar to the maximization problems discussed above the representative household maximizes lifetime utility.
- Regarding consumption c_t we consider a logarithmic period utility function which is concave.
- The diffference to the usual problem is that in addition we have “labor” n_t in the utility function.
- This means that the consumer also has to choose the optimal amount of labor she/he will supply.
- Hence, in addition to previous problems we have to differentiate with respect to n_t.
- What actually enters positively in the utility function is not labor but $(1 - n_t)$ which can be thought of leisure time.
- Labor enters negatively in the utility function.

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Solving the problem

- We substitute the production function into the capital accumulation equation and get the period budget constraint

$$k_{t+1} + c_t = (1-\delta)k_t + (a_t n_t)^\alpha k_t^{1-\alpha}. \tag{28}$$

- Again we use the Lagrangian to solve the problem

$$\mathcal{L} = \sum_{s=0}^{\infty} \beta^s \left[\ln(c_{t+s}) + \theta \frac{(1-n_{t+s})^{1-\gamma}}{1-\gamma} \right] + \cdots$$
$$\cdots \lambda_{t+s} \left[(a_{t+s} n_{t+s})^\alpha k_{t+s}^{1-\alpha} - k_{t+s+1} - c_{t+s} \right].$$

- The first order conditions are given by

$$\frac{\partial \mathcal{L}}{\partial c_{t+s}} = \beta^s \frac{1}{c_{t+s}} - \lambda_{t+s} \stackrel{!}{=} 0 \tag{I}$$

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Solving the problem

$$\frac{\partial \mathcal{L}}{\partial k_{t+s+1}} = \lambda_{t+s+1} \left[1 - \delta + (1+\alpha) \left(\frac{a_{t+s+1} n_{t+s+1}}{k_{t+s+1}} \right)^{\alpha} \right] - \lambda_{t+s} \stackrel{!}{=} 0 \quad \text{(II)}$$

$$\frac{\partial \mathcal{L}}{\partial n_{t+s}} = -\beta^{s} \left[\theta (1 - n_{t+s})^{-\gamma} \right] + \lambda_{t+s} \alpha a_{t+s}^{\alpha} \left(\frac{k_{t+s}}{n_{t+s}} \right)^{1-\alpha} \stackrel{!}{=} 0 \quad \text{(III)}$$

- Combining (I) with (II) and (III) yields

$$\frac{1}{c_{t+s}} = \beta \frac{1}{c_{t+s+1}} \left[1 - \delta + (1+\alpha) \left(\frac{a_{t+s+1} n_{t+s+1}}{k_{t+s+1}} \right)^{\alpha} \right] \quad (29)$$

$$\theta (1 - n_{t+s})^{-\gamma} = \frac{1}{c_{t+s}} \alpha a_{t+s}^{\alpha} \left(\frac{k_{t+s}}{n_{t+s}} \right)^{1-\alpha} . \quad (30)$$

Interpretation of the results

- Equation (29) is the usual Euler equation.
- The Euler equation is an intertemporal optimality condition.
- Since in the objective function we assumed that labor is additive seperable, labor does not influence the Euler equation, the interpretation stays the same as in previous problems.
- Equation (30) is an implicit expression for optimal labor supply of households.
- It is also independent of consumption because we have assumed additive seperability (instead of a multiplicative specification) in the lifetime utility function.
- Equation (30) determins how much labor households want to supply in a given period.
- Note that in contrast to (29) (30) is an intratemporal optimality condition.

The RBC model

- Having derived those optimality conditions and assuming a stochastic process for technology, one can set up the so-called *real business cycle (RBC) model*.
- In this model business cycles are generated by technology shocks alone.
- The RBC model is a dynamic stochastic general equilibrium model.
- This means that...
  - ... that it can describe a time path of variables
  - ... it has a stochastic component (technology shocks)
  - ... prices in the model (such as the interest rate) are determined by agents in the model
- This kind of models can explain basic business cycle facts such as volatility, correlations and autocorrelations.
- However, it has been found that technology shocks are not the source of business cycles.

The RBC model

- In addition to the households who maximize their lifetime utility by choosing their consumption path and labor supply, there are firms in the model who produce the consumption good.
- In the standard RBC-model equations are log-linearized.
- Then one can wirte a computer program to simulate the model.
- Analysis in this model is usually done by inspecting *impulse response functions (IRFs)* of main economic variables.
- In addition we can generate artificial data of output, the real interest rate, investment, consumption and labor.
- Descriptive statistics of those series will be close to the results we obtained by the HP-filter.

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