Notes on Technology

Starting at date 0 we draw independently from some fixed distribution at a Poisson rate $aR(s)$. By date t the number of draws is distributed Poisson with parameter aT where

$$T = \int_0^t R(s)ds.$$

Each draw i yields an efficiency parameter $Z_i \geq \underline{z}$ with

$$\Pr[Z_i \geq z] = \left(\frac{z}{\underline{z}}\right)^{-\theta}.$$

Thus, for any $z \geq \underline{z}$, the number of draws with $Z_i \geq z$ is Poisson with parameter

$$\lambda = aT\underline{z}^{\theta}z^{-\theta}.$$

We might as well set $a\underline{z}^{\theta} = 1$ and take the limit as $\underline{z} \longrightarrow 0$ (hence $a \longrightarrow \infty$), so that

$$\lambda = Tz^{-\theta}.$$

Now consider ordering the draws, not by when they were drawn, but by how good they are

$$Z^{(1)} > Z^{(2)} > Z^{(3)} > \ldots$$

We can characterize the joint distributions of the $Z^{(k)}$ quite easily. To do so, let $S^{(k)} = \left(Z^{(k)}\right)^{-\theta}$. The space of efficiency levels is mapped from $z \geq 0$ to $s = z^{-\theta} \geq 0$, with $z \longrightarrow \infty$ corresponding to $s = 0$.

We can characterize the stochastic process described above very simply as a Poisson process at intensity T on s, starting from $s = 0$. The spaces between the $S^{(k)}$ are given by

$$\tau^{(k)} = S^{(k)} - S^{(k-1)},$$

where we define $S^{(0)} = 0$. The $\tau^{(k)}$ form a sequence of i.i.d. draws from an exponential distribution

$$\Pr\left[\tau^{(k)} \leq x\right] = 1 - e^{-Tx}.$$

Thus

$$\Pr\left[S^{(1)} \leq s\right] = 1 - e^{-Ts}$$

and for any $k \geq 1$

$$\Pr\left[S^{(k+1)} \leq s_{k+1} | S^{(k)} = s_k\right] = 1 - e^{-T\left(s_{k+1} - s_k\right)}. \tag{1}$$

Let's check the results derived from this formulation. The distribution of the most efficient technique is

$$\begin{aligned}\Pr\left[Z^{(1)} \leq z\right] &= \Pr\left[S^{(1)} \geq z^{-\theta}\right] \\ &= e^{-Tz^{-\theta}},\end{aligned}$$

which is the Fréchet as in EK (2002). The joint distribution of the 1st and 2nd most efficient, for $0 \leq z_2 \leq z_1$, is

$$\begin{aligned}\Pr\left[Z^{(1)} \leq z_1, Z^{(2)} \leq z_2\right] &= \Pr\left[S^{(1)} \geq z_1^{-\theta}, S^{(2)} \geq z_2^{-\theta}\right] \\ &= \int_{z_1^{-\theta}}^{\infty} \Pr\left[S^{(2)} \geq z_2^{-\theta} | S^{(1)} = s_1\right] f(s_1) ds_1 \\ &= \int_{z_1^{-\theta}}^{z_2^{-\theta}} e^{-T(z_2^{-\theta}-s_1)} T e^{-Ts_1} ds_1 + \int_{z_2^{-\theta}}^{\infty} T e^{-Ts_1} ds_1 \\ &= \int_{z_1^{-\theta}}^{z_2^{-\theta}} T e^{-Tz_2^{-\theta}} ds_1 + e^{-Tz_2^{-\theta}} \\ &= \left[1 + T\left(z_2^{-\theta} - z_1^{-\theta}\right)\right] e^{-Tz_2^{-\theta}},\end{aligned}$$

where the third line follows because $S^{(2)} \geq S^{(1)}$ so that $S^{(2)}$ automatically exceeds $z_2^{-\theta}$ if $S^{(1)}$ does. Note that the resulting joint distribution is the same as in BEJK (2003).

The sum of k independent exponentials with parameter T is gamma with parameters k and T. Thus the density of $S^{(k)}$ is

$$f(s) = \frac{T^k}{(k-1)!} s^{k-1} e^{-Ts}$$

and the distribution function is

$$\begin{aligned} \Pr[S^{(k)} &\leq s_k] = \int_0^{s_k} \frac{T^k}{(k-1)!} s^{k-1} e^{-Ts} ds \\ &= \int_0^{Ts_k} \frac{1}{(k-1)!} x^{k-1} e^{-x} dx \\ &= 1 - \left(\sum_{i=1}^{k} \frac{(Ts_k)^{i-1}}{(i-1)!} \right) e^{-Ts_k}. \end{aligned}$$

Since

$$C^{(k)} = w/Z^{(k)} = w\left(S^{(k)}\right)^{1/\theta}$$

we have

$$\begin{aligned}
\Pr\left[C^{(k)} \leq c\right] &= \Pr\left[w\left(S^{(k)}\right)^{1/\theta} \leq c\right] = \Pr\left[S^{(k)} \leq \left(\frac{c}{w}\right)^{\theta}\right] \\
&= 1 - \left(\sum_{i=1}^{k} \frac{\left(T\left(\frac{c}{w}\right)^{\theta}\right)^{i-1}}{(i-1)!}\right) e^{-T\left(\frac{c}{w}\right)^{\theta}} \\
&= 1 - \left(\sum_{i=1}^{k} \frac{\left(\Phi c^{\theta}\right)^{i-1}}{(i-1)!}\right) e^{-\Phi c^{\theta}},
\end{aligned}$$

where $\Phi = Tw^{-\theta}$. This is the same distribution as in the EK book.

Finally, lets use (1) to derive the distribution of $C^{(k+1)}$ conditional on $C^{(k)}$:

$$\begin{aligned}
\Pr\left[C^{(k+1)} \leq c_{k+1} | C^{(k)} = c_k\right] &= \Pr\left[S^{(k+1)} \leq \left(\frac{c_{k+1}}{w}\right)^{\theta} | S^{(k)} = \left(\frac{c_k}{w}\right)^{\theta}\right] \\
&= 1 - e^{-T\left[\left(\frac{c_{k+1}}{w}\right)^{\theta} - \left(\frac{c_k}{w}\right)^{\theta}\right]} \\
&= 1 - e^{-\Phi\left[c_{k+1}^{\theta} - c_k^{\theta}\right]}.
\end{aligned}$$

Again, this is the same result as in the EK book.