1. **Utility maximization and budget constraints**

   Consider an objective function of the form

   $$V_t = \sum_{s=0}^{\infty} \beta^s U(c_{t+s}). \tag{1}$$

   Derive the Euler equation for consumption for each of the following ways of writing the budget constraint.

   (a) The budget constraint is given by

   $$a_{t+1} = (1+r)(a_t + x_t - c_t), \tag{2}$$

   i.e. current assets and income that is not consumed is invested.

   (b) The budget constraint is given by

   $$\Delta a_t + c_t = x_t + r a_{t-1}, \tag{3}$$

   where the dating convention is that a_t denotes the end of period stock of assets and c_t and x_t are consumption and income during period t.

   (c) The budget constraint is given by

   $$W_t = \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s c_{t+s} = \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s} + (1+r)a_t, \tag{4}$$

   where W_t is household wealth.

2. **Consumption smoothing**

   Households live for periods t and $t+1$. The discount factor for period $t+1$ is $\beta = 1$. They receive exogenous income x_t and x_{t+1}, where the conditional distribution of income in period $t+1$ is $\mathcal{N}(x_t, \sigma^2)$, but they have no assets.

   (a) Find a level of c_t that maximizes

   $$V_t = U(c_t) + \mathbb{E}_t U(c_{t+1}) \tag{5}$$

   if the utility function is quadratic

   $$U(c_t) = -\frac{1}{2}c_t^2 + \alpha c_t, \quad (\alpha > 0). \tag{6}$$

(b) Calculate the conditional variance of this level of c_t and comment on what this implies about consumption smoothing.

3. **Linearization**

Consider the general expression of a nonlinear difference equation

$$x_{t+1} = f(x_t), \tag{7}$$

where $f(\cdot)$ is potentially non-linear. Assume that the function has a steady state denoted by x^*.

(a) Linearize the function around the steady-state value x.
Hint: The first-order approximation you are asked to derive is given by the tangent of the function at the steady state.

(b) Illustrate your approximation with a graph. Use for example

$$x_{t+1} = f(x_t) = \sqrt{x_t}. \tag{8}$$

When is your approximation good, when is it bad?

(c) Now consider the same difference equation (7). Write the equation in percent deviations from the steady state using

$$\hat{x}_t \equiv \ln\left(\frac{x_t}{x}\right). \tag{9}$$

(d) What is the value of $\hat{x}$?

(e) Take the log-linear approximation of the general difference equation (7). You can use the function you derived under question (a) replacing x_t by $\hat{x}_t$.

(f) Now consider a Cobb-Douglas production function of the form

$$Y_t = F(K_t, L_t) = K_t^\alpha L_t^{1-\alpha}. \tag{10}$$

Log-linearize this equation. How would you call the coefficient on $\hat{k}_t$?