Solution sketch for Advanced Macro, Problem Set 1 (Golden rule):

For background: see Lecture 2, page 6-20
Assume:

$$y = k^{\alpha}, \qquad \text{with: } \alpha \in (0,1)$$

a) obvious (Lec 2, page 9)

b) obvious (Lec 2, page 8)

c) Notice

$$f'(k_{GR}) = \delta,$$

implying

$$\alpha k_{GR}^{\alpha-1} = \delta$$

Hence,

$$k_{GR} = (\frac{\delta}{\alpha})^{\frac{1}{\alpha-1}}$$

For further use in d) below calculation of c_{GR} :

$$\begin{aligned}
c_{GR} &= f(k_{GR}) - \delta k_{GR} \\
&= (\frac{\delta}{\alpha})^{\frac{\alpha}{\alpha-1}} - \delta \cdot (\frac{\delta}{\alpha})^{\frac{1}{\alpha-1}} \\
&= \delta^{\frac{\alpha}{\alpha-1}} \cdot \left((\frac{1}{\alpha})^{\frac{\alpha}{\alpha-1}} - (\frac{1}{\alpha})^{\frac{1}{\alpha-1}} \right) \\
&= \delta^{\frac{\alpha}{\alpha-1}} \cdot (\frac{1}{\alpha})^{\frac{\alpha}{\alpha-1}} \cdot \left(1 - (\frac{1}{\alpha})^{-1} \right) \\
&= \delta^{\frac{\alpha}{\alpha-1}} \cdot (\frac{1}{\alpha})^{\frac{\alpha}{\alpha-1}} \cdot (1-\alpha)
\end{aligned}$$

d) Verify the comparative statics results with respect to δ derived in the Lecture, ie

$$\text{(i) } \frac{\partial k_{GR}}{\partial \delta} = \frac{1}{f''(k_{GR})} \quad \text{and} \quad \text{(ii) } \frac{\partial c_{GR}}{\partial \delta} = -k_{GR}$$

$\rightarrow$ **ad part (i):**

$$\begin{aligned}
\frac{\partial k_{GR}}{\partial \delta} &= \frac{1}{\alpha-1} \cdot (\frac{1}{\alpha})^{\frac{1}{\alpha-1}} \cdot \delta^{\frac{1}{\alpha-1}-1} \\
&= \underbrace{\frac{1}{\alpha-1}}_{<0} \cdot \underbrace{(\frac{1}{\alpha})^{\frac{1}{\alpha-1}}}_{>0} \cdot \underbrace{\delta^{\frac{2-\alpha}{\alpha-1}}}_{>0} < 0
\end{aligned}$$

Since

$$f''(k) = \alpha(\alpha-1)k^{\alpha-2}$$

we get

$$\begin{aligned}
f''(k_{GR}) &= \alpha(\alpha-1)\left((\frac{\delta}{\alpha})^{\frac{1}{\alpha-1}}\right)^{\alpha-2} \\
&= \alpha(\alpha-1)(\frac{\delta}{\alpha})^{\frac{\alpha-2}{\alpha-1}} \\
&= (\alpha-1)\cdot\alpha^{1-\frac{\alpha-2}{\alpha-1}}\cdot\delta^{\frac{\alpha-2}{\alpha-1}} \\
&= (\alpha-1)\cdot\alpha^{\frac{1}{\alpha-1}}\cdot\delta^{\frac{\alpha-2}{\alpha-1}}
\end{aligned}$$

Hence,

$$\frac{\partial k_{GR}}{\partial \delta} = \frac{1}{f''(k_{GR})}$$

→ **ad part (ii):**

$$c_{GR} = \delta^{\frac{\alpha}{\alpha-1}}\cdot(\frac{1}{\alpha})^{\frac{\alpha}{\alpha-1}}\cdot(1-\alpha)$$

$$\begin{aligned}
\frac{\partial c_{GR}}{\partial \delta} &= \frac{\alpha}{\alpha-1}\cdot\delta^{\frac{\alpha}{\alpha-1}-1}\cdot(\frac{1}{\alpha})^{\frac{\alpha}{\alpha-1}}\cdot(1-\alpha) \\
&= -\alpha\cdot(\frac{1}{\alpha})^{\frac{\alpha}{\alpha-1}}\cdot\delta^{\frac{1}{\alpha-1}} \\
&= -\alpha^{-\frac{1}{\alpha-1}}\cdot\delta^{\frac{1}{\alpha-1}} \\
&= -(\frac{\delta}{\alpha})^{\frac{1}{\alpha-1}}
\end{aligned}$$

Hence,

$$\frac{\partial c_{GR}}{\partial \delta} = -k_{GR}$$

Upshot:

→ although Cobb-Douglas is a standard assumption it turns out that the detailed derivations for the comparative statics exercise for this particular production function are rather algebra-intensiv (ie 'unpleasant').

→ Compare this with the short derivations on slides 18-20 under the general neoclassical assumptions (see slide 8/9) that were made with respect to the properties of $f(k)$

→ Hence, often it is advantageous to stick to general assumptions, ie not to turn too early to seemingly simple specific functional forms