Advanced Macroeconomics

The Solow growth model

Günter W. Beck

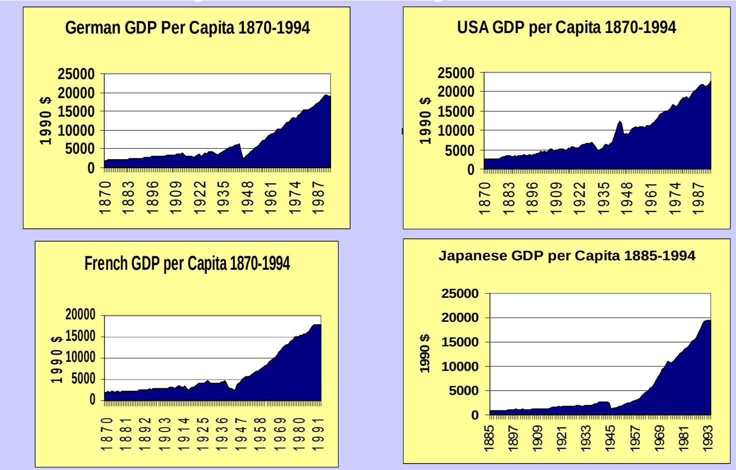
University of Mainz

November 23, 2010

Overview

Overview

GDP per capita of major industrialized countries



Growth rates of major industrialized countries

Country	Level of GDP per capita in 1870 (in 1985 dollars)	Level of GDP per capita in 1996 (in 1985 dollars)	Annual Growth rate 1870-1996
USA	2,247	19,638	1.7%
UK	2,610	14,440	1.4%
Canada	1,347	17,453	2.1%
Australia	3,123	15,076	1.3%
Japan	618	17,346	2.7%
Germany	1,300	15,313	2.0%

Implications of different growth rates

What if ...?

If US growth 1% lower since 1870 then current output per capita would equal Mexico's.

if US had grown as fast as Japan over this period US output 3.3 times higher.

If Ethiopia grows as fast as US since 1870 reach current US standards in 239 years

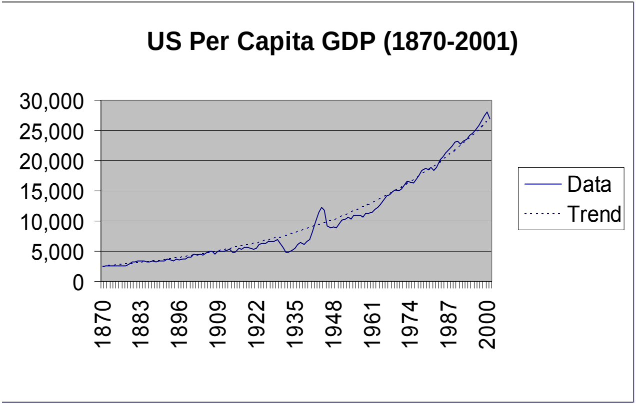
If Ethiopia grows as fast as Japan since 1870 reach US standards in 152 years

Relatively small changes have enormous long run impact.

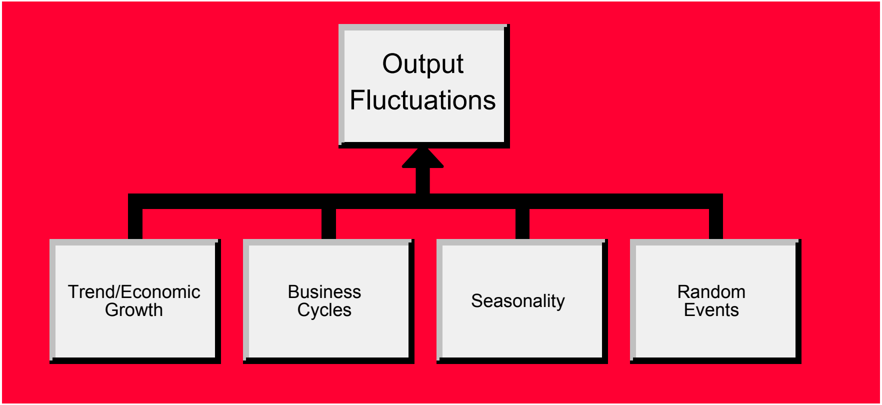
Stylized facts of economic growth

- In the short run, important fluctuations: Output, employment, investment, and consumption vary a lot across booms and recessions.
- In the long run, balanced growth: Output per worker and capital per worker (Y/L and K/L) grow at roughly constant, and certainly not vanishing, rates.
- The capital-to-output ratio (K/Y) is nearly constant.
- The return to capital (r) is roughly constant, whereas the wage rate (w) grows at the same rates as output.
- The income shares of labor and capital (wL/Y and rK/Y) stay roughly constant.
- Substantial cross-country differences in both income levels and growth rates.

Decomposition of GDP series 1



Decomposition of GDP series 2



Overview

Model assumptions (variables)

- **Production function:**

$$Y_t = F\left(K_t, A_t L_t\right), \tag{1}$$

with

- Y: output
- K: capital input
- L: labor input
- A: level of technology, knowledge, efficiency of work
- AL: effective labor

Model assumptions (functions) 1

- **Assumptions concerning the production function:**
  - Constant returns to scale:

$$F(cK, cAL) = cF(K, AL) \quad \text{for all } c \geq 0. \tag{2}$$

  - Positive, but declining marginal products of capital and labor

$$\frac{\partial F(\bullet)}{\partial K} > 0 \text{ and } \frac{\partial^2 F(\bullet)}{\partial K \partial K} < 0 \tag{3}$$

and

$$\frac{\partial F(\bullet)}{\partial L} > 0 \text{ and } \frac{\partial^2 F(\bullet)}{\partial L \partial L} < 0 \tag{4}$$

Model assumptions (functions) 2

- **Assumptions concerning the production function:**
  - Both production factors are necessary

$$F(0, AL) = 0 \text{ and } F(K, A0) = 0 \tag{5}$$

  - Inada conditions are satisfied

$$\lim_{K \to 0} \frac{\partial F(\bullet)}{\partial K} \to \infty \text{ and } \lim_{K \to \infty} \frac{\partial F(\bullet)}{\partial K} = 0 \tag{6}$$

and

$$\lim_{L \to 0} \frac{\partial F(\bullet)}{\partial L} \to \infty \text{ and } \lim_{L \to \infty} \frac{\partial F(\bullet)}{\partial L} = 0 \tag{7}$$

The per capita production function

- The production function in intensive form:
  - Define $y = \frac{Y}{AL}$ as output per unit of effective labor and $k = \frac{K}{AL}$ as capital per unit of effective labor.
  - Then the production function can be transformed as follows:

$$y = \frac{Y}{AL} = \frac{F(K, AL)}{AL} = f(k), \tag{8}$$

where $f(k) = F(k, 1)$.

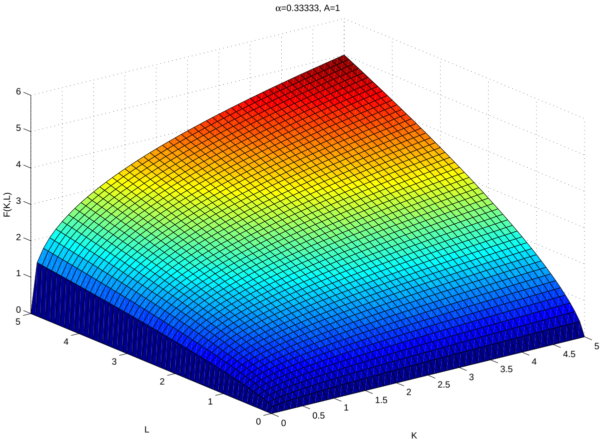
The Cobb-Douglas production function

- Special case: Cobb-Douglas production

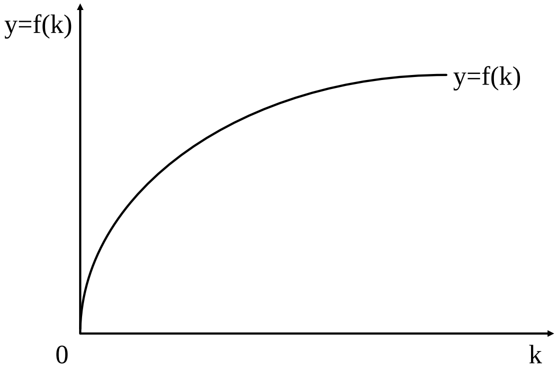
$$F(K, AL) = K^{\alpha}(AL)^{1-\alpha}, \tag{9}$$

where $0 < \alpha < 1$.

Graphical illustration of a CD production function



The CD production function in intensive form



Exogenous growth of labor and technology

- The evolution of the production input factors
  - Labor

$$\dot{L}_t = nL_t, \tag{10}$$

where $\dot{L}_t = \frac{\partial L_t}{\partial t}$ (in other words: n is the growth rate of labor).
  - Technology

$$\dot{A}_t = gA_t. \tag{11}$$

  - Note that this implies (differential equation)

$$A_t = A_0 \mathrm{e}^{gt}, \quad L_t = L_0 \mathrm{e}^{nt} \tag{12}$$

- Further assumptions:
  - Saving rate, s, is exogenous.
  - Depreciation rate, δ, is exogenous.
  - Economy is closed, i.e., aggregate savings are equal to aggregate investemt ($S = I$).

Overview

Capital accumulation

- **The dynamics of K**

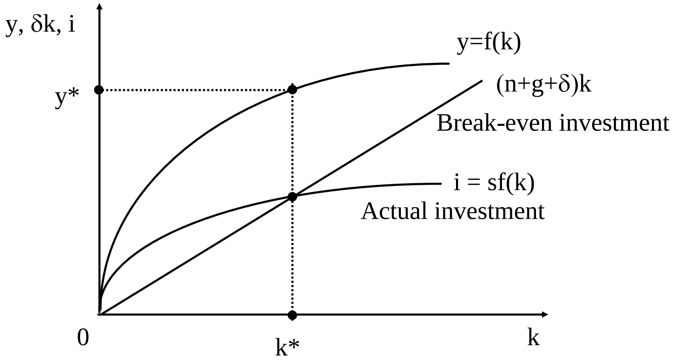
$$\dot{K}_t = sY_t - \delta K_t \tag{13}$$

- **The dynamics of k**
  - Remember: k_t is defined as $\frac{K_t}{A_t L_t}$.
  - Differentiating this expression with respect to t, observing that K_t, A_t and L_t all depend on time, and using the product, quotient and chain rule of differentiation, one obtains (please check):

$$\dot{k}_t = sf(k_t) - (n + g + \delta)k_t \tag{14}$$

- The above equation states that the rate of change of k_t is the difference between
  - **actual investment per unit of effective labor** $(sf(k_t)$ and
  - **break-even investment** $((n + g + \delta)k_t)$ (amount of investment necessary to keep k at its existing level).

The dynamics of the economy 1



The dynamics of the economy 2

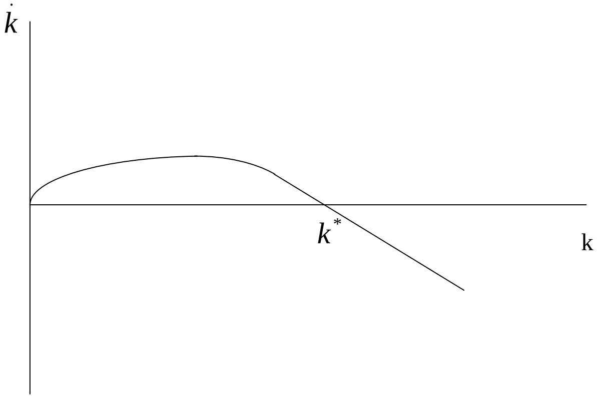


Figure 1.3 The phase diagram for k in the Solow model

The balanced growth path

- Balanced growth path (steady state): A situation where each variable is growing at a constant rate.
- In equilibrium
  - L grows at rate n
  - A grows at rate g
  - K grows at rate $n+g$
  - Y grows at rate $n+g$
  - $\frac{K}{L}$ grows at rate g
  - $\frac{Y}{L}$ grows at rate g
- A nice way to show some of the results above is to take logs and differentiate with respect to time t.
- Note that in general we can compute growth rates as follows

$$\frac{\partial \ln x}{\partial t} = \frac{1}{x} \cdot \dot{x}_t = \frac{\dot{x}_t}{x_t} = g_x.$$

The impact of a change in the saving rate

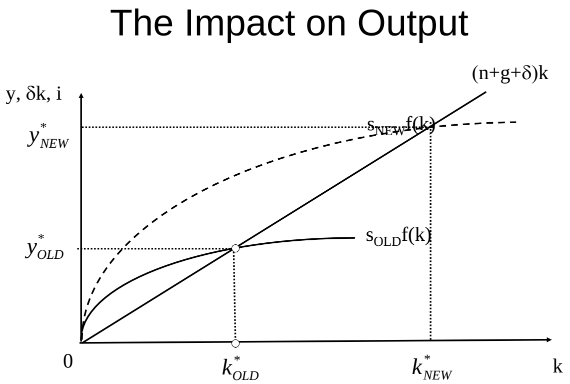


Figure 1.4 The effects of an increase in the saving rate on investment

The impact of Δs on consumption 1

- An increase in the saving rate has the following two effects on consumption (Remember: Steady-state consumption is given by $c^* = (1-s)f(k^*)$, where an * denotes steady-state values.):
  - Initially, the increase in s lowers consumption.
  - However, in the medium- and long-run the initial effect is mitigated through the increase in k. In the new steady state consumption might even be higher.
- Algebraically, the impact of a change in s can be computed as follows:
- In the steady state we have:

$$c^* = f(k^*) - sf(k^*) = f(k^*) - (n+g+\delta)k^*, \tag{15}$$

with $k^* = k^*(s, n, g, \delta)$.

The impact of Δs on consumption 2

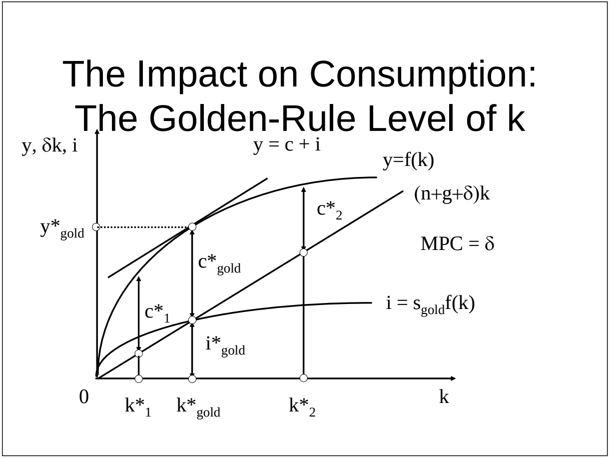
- To see what impact a change in s has on c^* this expression has to be differentiated with respect to s. The result is:

$$\frac{\partial c^*}{\partial s} = \left[f'(k^*(s,n,g,\delta)) - (n+g+\delta) \right] \frac{\partial k^*(s,n,g,\delta)}{\partial s}. \qquad (16)$$

- The second term of this expression (derivative of k with respect to s) is always positive. (Why?)
- Thus, a change in s increases steady-state consumption (per unit of effective labor) if the expression in the squared brackets is larger than zero, given the old steady-state value of k.
- Maximum steady-state consumption is reached when $f'(k^*(s,n,g,\delta)) = (n+g+\delta)$. In this case, k has reached its "golden-rule level".

Overview

Golden-rule level of k



The speed of convergence 1

- Objective: Want to know how fast k converges to steady state.
- Solution: Obtain solution of differential equation for k derived above (see equation (14)).
- Assume that production function is Cobb-Douglas. Then:

$$\dot{k}_t = sk_t^{\alpha} - (n + g + \delta)k_t. \tag{17}$$

- Problem: Equation is nonlinear.
- Trick: Define the capital/output ratio, denoted by x_t, as follows:

$$x_t = \frac{k_t}{y_t} = \frac{k_t}{k_t^{\alpha}} = k_t^{1-\alpha}. \tag{18}$$

- Then (please check):

$$\dot{x}_t = (1 - \alpha)k_t^{-\alpha}\dot{k}_t \tag{19}$$

The speed of convergence 2

and

$$\frac{\dot{x}_t}{x_t} = \frac{(1-\alpha)k_t^{-\alpha}\dot{k}_t}{k_t^{1-\alpha}} = (1-\alpha)\frac{\dot{k}_t}{k_t} \tag{20}$$

- Transform the differential equation for $\dot{k}$ as follows:

$$\frac{\dot{k}_t}{k_t} = sk_t^{\alpha-1} - (n+g+\delta) \tag{21}$$

- Solving equation (20) for $\frac{\dot{k}_t}{k_t}$ and plugging the resulting expression into (21) yields:

$$\frac{1}{1-\alpha}\frac{\dot{x}_t}{x_t} = \frac{s}{x_t} - (n+g+\delta). \tag{22}$$

The steady state

- This can be rewritten as:

$$\dot{x}_t = (1-\alpha)s - (1-\alpha)(n+g+\delta)x_t. \tag{23}$$

This is a linear differential equations with constant coefficients and a constant forcing term. To solve it we proceed as follows:

- First, we compute the steady state as follows

$$x^* = \frac{s}{n+g+\delta} \tag{24}$$

Transforming the equation

- Secondly, we the transform the differential equation into a homogenous equation. To do so introduce a new variable, z_t, which is defined as the difference between the actual capital/output ratio and its steady-state value, i.e.,

$$z_t \equiv x_t - x^*. \tag{25}$$

- Then

$$\begin{aligned} \dot{z}_t = \dot{x}_t &= (1-\alpha)s - (1-\alpha)(n+g+\delta)x_t \qquad (26) \\ &= (1-\alpha)s - (1-\alpha)(n+g+\delta)\left[z_t + x^*\right] \\ &= (1-\alpha)s - (1-\alpha)(n+g+\delta)\left[z_t + \frac{s}{n+g+\delta}\right] \\ &= -(1-\alpha)(n+g+\delta)z_t \end{aligned}$$

Solution of the equation

- Defining $\lambda \equiv (1-\alpha)(n+g+\delta)$ this equation can be rewritten as

$$\dot{z}_t = -\lambda z_t \tag{27}$$

- Thirdly, solve the transformed homogenous differential equation as follows:

$$z_t = e^{-\lambda t} z_0. \tag{28}$$

- Plugging back in the definition of z_t and rearranging gives:

$$x_t = \left(1 - e^{-\lambda t}\right) x^* + e^{-\lambda t} x_0. \tag{29}$$

Properties of the solution

The actual capital/output ratio is a weighted average of its steady-state value.

- Note that:

$$\lim_{t\to\infty} e^{-\lambda t} = 0. \tag{30}$$

- The speed at which convergence to the steady state occurs is measured by $\lambda = (1-\alpha)(n+g+\delta)$. Higher values of α, e.g., imply lower speeds of convergence.
- Since $x_t = k_t^{1-\alpha}$ we obtain for k_t:

$$k_t = \left[\left(1-e^{-\lambda t}\right)\frac{s}{n+g+\delta} + e^{-\lambda t}k_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}. \tag{31}$$