GSEFM
Graduate School of Economics, Finance and Management

Phd Programme

Mathematical Methods
Part II

Lecturer:

Prof. Dr. Klaus Wälde
Universität Mainz

Tutorial:

Michael Lamprecht
Universität Mainz

www.waelde.com/aio
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Mathematical Methods
Exercises to chapter 2

for all questions, please contact Michael Lamprecht at lamprecht@uni-mainz.de

Capital Market Restrictions – Ex. 3

Now consider the following budget constraint. This is a budget constraint that would be appropriate if you want to study the education decisions of households. The parameter b amounts to schooling costs. Inheritance of this individual under consideration is n.

$$U_t = \gamma \ln c_t + (1 - \gamma) \ln c_{t+1}$$

subject to

$$-b + n + (1 + r)^{-1} w_{t+1} = c_t + (1 + r)^{-1} c_{t+1}.$$

(a) What is the optimal consumption profile under no capital market restrictions?

(b) Assume loans for financing education are not available, hence savings need to be positive, $s_t \geq 0$. What is the consumption profile in this case?

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The Envelope Theorem I – Ex. 1

Let the utility function of an individual be given by

$$U = U(C, L),$$

where consumption C increases utility and supply of labour L decreases utility. Let the budget constraint of the individual be given by

$$wL = C.$$

Let the individual maximize utility with respect to consumption and the amount of labour supplied.

(a) What is the optimal labour supply function (in implicit form)? How much does an individual consume? What is the indirect utility function?

(b) Under what conditions does an individual increase labour supply when wages rise (no analytical solution required)?

(c) Assume higher wages lead to increased labour supply. Does disutility arising from increased labour supply compensate utility from higher consumption? Does utility rise if there is no disutility from working? Start from the indirect utility function derived in a) and apply the proof of the envelope theorem and the envelope theorem itself.

Environmental Economics – Ex. 8

Imagine you are an economist only interested in maximizing the present value of your endowment. You own a renewable resource, for example a piece of forest. The amount of wood in your forest at a point in time t is given by x_t. Trees grow at $b(x_t)$ and you harvest at t the quantity c_t.

(a) What is the law of motion for x_t?

(b) What is your objective function if prices at t per unit of wood is given by p_t, your horizon is infinity and you have perfect information?

(c) How much should you harvest per period when the interest rate is constant? Does this change when the interest rate is time-variable?

The 10k run - Formulating and Solving a Maximization Problem – Ex. 9

You consider participation in a 10k run or a marathon. The event will take place in M months. You know that your fitness needs to be improved and that this will be costly: it requires effort a_0 which reduces utility $u(.)$. At the same time, you enjoy being fast, i.e. utility increases the shorter your finish time l. The higher your effort, the shorter your finish time.

(a) Formulate a maximization problem with 2 periods. Effort affects the finish time in M months. Specify a utility function and discuss a reasonable functional form which captures the link between finish time l and effort a_0.

(b) Solve this maximization problem by providing and discussing the first-order condition.

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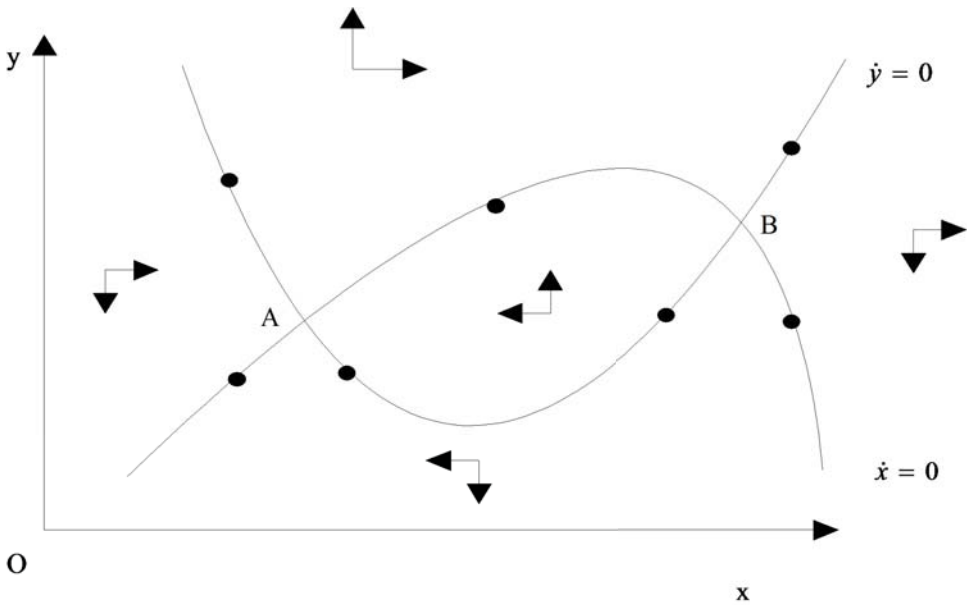
Phase Diagram III - Ex. 3

(a) Plot paths through points marked by a dot "·" in the figure below.

(b) What type of fixpoints are A and B?



Phase Diagram and Fixpoint – Ex. 5

Grossman and Helpman (1991) present a growth model with an increasing number of varieties. The reduced form of this economy can be described by a two-dimensional differential equation system,

$$\dot{n}(t) = \frac{L}{a} - \frac{\alpha}{v(t)}, \quad \dot{v}(t) = \rho v(t) - \frac{1-\alpha}{n(t)},$$

where $0 < \alpha < 1$ and $a > 0$. Variables $v(t)$ and $n(t)$ denote the value of the representative firm and the number of firms, respectively. The positive constants ρ and L denote the time preference rate and fix labour supply.

(a) Draw a phase diagram (for positive $n(t)$ and $v(t)$) and determine the fixpoint.

(b) What type of fixpoint do you find?

Comparing Forward and Backward Solution – Ex. 7

Remember that $\int_{z_1}^{z_2} f(z)\,dz = -\int_{z_2}^{z_1} f(z)\,dz$ for any well-defined $z_1,\ z_2$ and $f(z)$. Replace T by t_0 in (4.3.8) and show that the solution is identical to the one in (4.3.7). Explain why this must be the case.

A Budget Constraint with many Assets – Ex. 10

Consider an economy with two assets whose prices are $v_i(t)$. A household owns $n_i(t)$ assets of each type such that total wealth at time t of the household is given by $a(t) = v_1(t) n_1(t) + v_2(t) n_2(t)$. Each asset pays a flow of dividends $\pi_i(t)$. Let the household earn wage income $w(t)$ and spend $p(t) c(t)$ on consumption per unit of time. Show that the household's budget constraint is given by

$$\dot{a}(t) = r(t) a(t) + w(t) - p(t) c(t)$$

where the interest rates are defined by

$$r(t) \equiv \theta(t) r_1(t) + (1 - \theta(t)) r_2(t), \qquad r_i(t) \equiv \frac{\pi_i(t) + \dot{v}_i(t)}{v_i(t)}$$

and $\theta(t) \equiv v_1(t) n_1(t) / a(t)$ is defined as the share of wealth held in asset 1.

Optimal Saving – Ex. 11

Let optimal saving and consumption behaviour (see ch. 5, e.g. eq. $(5.1.6)$) be described by the two-dimensional system

$$\dot{c} = gc, \quad \dot{a} = ra + w - c,$$

where g is the growth rate of consumption, given e.g. by $g = r - \rho$ or $g = (r - \rho)/\sigma$. Solve this system for time paths of consumption c and wealth a.

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Consumption over the Life Cycle – Ex. 3

The utility of an individual, born at s and living for T periods is given at time t by

$$u(s,t) = \int_t^{s+T} e^{-\rho[\tau-t]} \ln(c(s,\tau))\, d\tau.$$

The individual's budget constraint is given by

$$\int_t^{s+T} D_R(\tau)\, c(s,\tau)\, d\tau = h(s,t) + a(s,t)$$

where

$$D_R(\tau) = \exp\left[-\int_t^{\tau} r(u)\, du\right], \quad h(s,t) = \int_t^{s+T} D_r(\tau)\, w(s,\tau)\, d\tau.$$

This deplorable individual would like to know how he can lead a happy life but, unfortunately, has not studied optimal control theory!

(a) What would you recommend him? Use a Hamiltonian approach and distinguish between changes of consumption and the initial level. Which information do you need to determine the initial consumption level? What information would you expect this individual to provide you with? In other words, which of the above maximization problems makes sense? Why not the other one?

(b) Assume all prices are constant. Draw the path of consumption in a $(t, c(t))$ diagram. Draw the path of asset holdings $a(t)$ in the same diagram, by guessing how you would expect it to look. (You could compute it if you want)

Optimal Consumption Levels – Ex. 6

(a) Derive a rule for the optimal consumption level for a time-varying interest rate $r(t)$. Show that $(5.6.4)$ can be generalized to

$$c(t) = \frac{1}{\int_t^\infty e^{-\int_t^\tau \frac{\rho-(1-\sigma)r(s)}{\sigma} ds} d\tau} \{W(t) + a(t)\},$$

where $W(t)$ is human wealth.

(b) What does this imply for the wealth level $a(t)$?

An Exam Question – Ex. 9

Consider a decentralized economy in continuous time. Factors of production are capital and labour. The initial capital stock is K_0, labour endowment is L. Capital is the only asset, i.e. households can save only by buying capital. Capital can be accumulated also at the aggregate level, $\dot{K}(\tau) = I(\tau) - \delta K(\tau)$. Households have a corresponding budget constraint and a standard intertemporal utility function with time preference rate ρ and infinite planning horizon. Firms produce under perfect competition. Describe such an economy in a formal way and derive its reduced form. Do this step by step:

(a) Choose a typical production function.

(b) Derive factor demand functions by firms.

(c) Let the budget constraints of households be given by

$$\dot{a}(\tau) = r(\tau) a(\tau) + w^L(\tau) - c(\tau).$$

Specify the maximization problem of a household and solve it.

(d) Aggregate the optimal individual decision over all households and describe the evolution of aggregate consumption.

(e) Formulate the goods market equilibrium.

(f) Show that the budget constraint of the household is consistent with the aggregate goods market equilibrium.

(g) Derive the reduced form

$$\dot{K}(\tau) = Y(\tau) - C(\tau) - \delta K(\tau), \quad \frac{\dot{C}(\tau)}{C(\tau)} = \frac{\frac{\partial Y(\tau)}{\partial K(\tau)} - \delta - \rho}{\sigma}$$

by going through these steps and explain the economics behind this reduced form.

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A Firm with Adjustment Costs – Ex. 2

Consider again, as in ch. 5.5.1, a firm with adjustment cost. The firm's objective is

$$\max_{\{I(t),L(t)\}} \int_t^\infty e^{-r[\tau-t]}\pi(\tau)\, d\tau.$$

In contrast to ch. 5.5.1, the firm now has an infinite planning horizon and employs two factors of production, capital and labour. Instantaneous profits are

$$\pi = pF(K,L) - wL - I - \alpha I^\beta,$$

where investment I also comprises adjustment costs for $\alpha > 0$. Capital, owned by the firm, accumulates according to $\dot{K} = I - \delta K$. All parameters δ, α, β are constant.

(a) Solve this maximization problem by using the dynamic programming approach. You may choose appropriate (numerical or other) values for parameters where this simplifies the solution (and does not destroy the spirit of this exercise).

(b) Show that in the long-run with adjustment costs and at each point in time under the absence of adjustment costs, capital is paid its value marginal product. Why is labour being paid its value marginal product at each point in time?

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Exercises to chapter 7 and 8

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Stochastic Difference Equations – Ex. 4

Consider the following stochastic difference equation

$$y_{t+1} = by_t + m + v_{t+1}, \quad v_t \sim N(0, \sigma_v^2)$$

(a) Describe the limiting distribution of y_t.

(b) Does the expected value of y_t converge to its fixpoint monotonically? How does the variance of y_t evolve over time?

Saving under Uncertainty – Ex. 5

Consider an individual's maximization problem

$$\max E_t \left\{ u\left(c_t\right) + \beta u\left(c_{t+1}\right) \right\}$$
$$\text{subject to } w_t = c_t + s_t, \quad \left(1 + r_{t+1}\right) s_t = c_{t+1}$$

(a) Solve this problem by replacing her second period consumption by an expression that depends on first period consumption.

(b) Consider now the individual's decision problem given the utility function $u\left(c_t\right) = c_t^\sigma$. Should you assume a parameter restriction on σ?

(c) What is your implicit assumption about β? Can it be negative or larger than one? Can the time preference rate ρ, where $\beta = \left(1 + \rho\right)^{-1}$, be negative?

OLG in General Equilibrium – Ex. 7

Build an OLG model in general equilibrium with capital accumulation and auto-regressive total factor productivity, $\ln A_{t+1} = \gamma \ln A_t + \varepsilon_{t+1}$ with $\varepsilon_{t+1} \sim N(\varepsilon, \sigma^2)$. What is the reduced form? Can a phase-diagram be drawn?

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Endogenous Labor Supply – Ex. 3

Solve the endogenous labour supply setup in ch. 9.4 by using dynamic programming.

Habit Formation – Ex. 5

Assume instantaneous utility depends, not only on current consumption, but also on habits (see for example Abel, 1990). Let the utility function therefore look like

$$U_t = E_t \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u\left(c_\tau, v_\tau\right),$$

where v_τ stands for habits like e.g. past consumption, $v_\tau = v\left(c_{\tau-1}, c_{\tau-2}, ...\right)$. Let such an individual maximize utility subject to the budget constraint

$$a_{t+1} = \left(1 + r_t\right) a_t + w_t - p_t c_t$$

(a) Assume the individual lives in a deterministic world and derive a rule for an optimal consumption path where the effect of habits are explicitly taken into account. Specify habits by $v_\tau = c_{\tau-1}$.

(b) Let there be uncertainty with respect to future prices. At a point in time t, all variables indexed by t are known. What is the optimal consumption rule when habits are treated in a parametric way?

(c) Choose a plausible instantaneous utility function and discuss the implications for optimal consumption given habits $v_\tau = c_{\tau-1}$.

Matching on Labour Markets – Ex. 9

Let employment L_t in a firm follow

$$L_{t+1} = (1-s)\,L_t + \mu V_t,$$

where s is a constant separation rate, μ is a constant matching rate and V_t denotes the number of jobs a firm currently offers. The firm's profits π_τ in period τ are given by the difference between revenue $p_\tau Y(L_\tau)$ and costs, where costs stem from wage payments and costs for vacancies V_τ captured by a parameter γ,

$$\pi_\tau = p_\tau Y(L_\tau) - w_\tau L_\tau - \gamma V_\tau.$$

The firm's objective function is given by

$$\Pi_t = E_t \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} \pi_\tau,$$

where β is a discount factor and E_t is the expectations operator.

(a) Assume a deterministic world. Let the firm choose the number of vacancies optimally. Use a Lagrangian to derive the optimality condition. Assume that there is an interior solution. Why is this an assumption that might not always be satisfied from the perspective of a single firm?

(b) Let us now assume that there is uncertain demand which translates into uncertain prices p_τ which are exogenous to the firm. Solve the optimal choice of the firm by inserting all equations into the objective function. Maximize by choosing the state variable and explain also in words what you do. Give an interpretation of the optimality condition. What does it imply for the optimal choice of V_τ?

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Exercises to chapter 10

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Expectations – Ex. 1

(a) Assume the price of bread follows a geometric Brownian motion. What is the probability that the price will be 20% more expensive in the next year?

(b) Consider a Poisson process with arrival rate λ. What is the probability that a jump occurs only after 3 weeks? What is the probability that 5 jumps will have occurred over the next 2 days?

Option Pricing – Ex. 6

Assume the price of an asset follows $dS/S = \alpha dt + \sigma dz + \beta dq$ (as in Merton, 1976), where z is Brownian motion and q is a Poisson process. This is a generalization of (10.3.1) where $\beta = 0$. How does the differential equation look like that determines the price of an option on this asset?

Martingales – Ex. 7

(a) The weather tomorrow will be just the same as today. Is this a martingale?

(b) Let $z(s)$ be Brownian motion. Show that $Y(t)$ defined by

$$Y(t) \equiv \exp\left[-\int_0^t f(s)\,dz(s) - \frac{1}{2}\int_0^t f^2(s)\,ds\right] \tag{1}$$

is a martingale.

(c) Show that $X(t)$ defined by

$$X(t) \equiv \exp\left[\int_0^t f(s)\,dz(s) - \frac{1}{2}\int_0^t f^2(s)\,ds\right]$$

is also a martingale.

Expected Returns – Ex. 10

Consider the budget constraint

$$da\left(t\right)=\left\{ra\left(t\right)+w-c\left(t\right)\right\}dt+\beta a\left(t\right)dz\left(t\right).$$

(a) What is the expected return for wealth? Why does this expression differ from (10.5.13)?

(b) What is the variance of wealth?

Non-Wage Labor Costs – Ex. 14

We consider a firm which can hire or fire workers at each instant $t \geq 0$. Each worker gets a direct wage compensation $w(t)$ in t. But there are also non-wage labor costs the firm has to pay at each point in time t. The ratio of these costs to the stochastic wage-related costs is $\tau(t)$. The process $\{w(t)\}$ is described by the stochastic differential equation

$$dw(t) = \sigma w(t) dW(t) - \eta \frac{w(t)}{\tau(t)} d\tau(t) \tag{2}$$

where $\{W(t)\}$ is a Brownian motion with mean 0 and variance t. We assume that the non-wage labor costs are completely described by a payroll-tax which is identical to the ratio $\tau(t)$. Because of political uncertainty the payroll-taxes $\tau(t)$ can jump at each instant t. This type of uncertainty is represented by the stochastic differential equation

$$d\tau(t) = \phi_1 \tau(t) dq_1(t) - \phi_2 \tau(t) dq_2(t), \qquad \phi_1, \phi_2 \geq 0 \tag{3}$$

with $\{q_1(t)\}$ and $\{q_2(t)\}$ are independent Poisson processes with arrival rates $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, respectively. The value function of the firm is $V(w(t), \tau(t))$ in t and has $w(t)$ and $\tau(t)$ as their arguments.

(a) How does the value function $V(.)$ evolve over time if there is no political uncertainty, i. e. $\tau(t) = \tau_0$, $\tau_0 \in [0, 1]$ for all $t \geq 0$?

(b) How does the value function $V(.)$ evolve over time if the direct wage costs are constant, i. e. $w(t) = w_0$, $w_0 \geq 0$ for all $t \geq 0$?

(c) Let us assume there is no fluctuation in the direct wages and there is no change in the payroll tax in time. How does the value function $V(.)$ evolve over time?

(d) Let us assume that the Poisson processes $\{q_1(t)\}$, $\{q_2(t)\}$ and $\{W(t)\}$ are independent. How does the value function $V(.)$ evolve over time if the direct wage costs $w(t)$ follows the stochastic differential equation (2) and there is political uncertainty?

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Standard and Non-Standard Technologies – Ex. 7

Let the social welfare function of a central planner be given by

$$U(t) = E_t \int_t^\infty e^{-\rho(\tau - t)} u\left(C(\tau)\right) d\tau$$

(a) Consider an economy where the capital stock follows $dK = AK^\alpha L^{1-\alpha}\left[\mu dt + \sigma dz\right] - (\delta K + C)\, dt$ where dz is the increment of Brownian motion and μ and σ are constants. Derive the Keynes-Ramsey rule for this economy.

(b) Assume that $dY = \Theta\left[\mu dt + \sigma dz\right]$ and that Θ is constant. What is the expected level of $Y(\tau)$ for $\tau > t$, i.e. $E_t Y(\tau)$?

(c) Consider an economy where the technology is given by $Y = AK^\alpha L^{1-\alpha}$ with $dA = \gamma A dt + \beta A dz$, where z is Brownian motion. Let the capital stock follow $dK = (Y - \delta K - C)\, dt$. Derive the Keynes-Ramsey rule for this economy as well.

(d) Is there a parameter constellation under which the Keynes-Ramsey rules are identical?

Standard and Non-Standard Technologies – Ex. 8

Provide answers to the same questions as in "Standard and non-standard technologies" but assume that z is a Poisson process with arrival rate λ. Compare your result to (11.5.6).

Finite Lives – Ex. 9

Households live a life which has potentially an infinite length. However, there is a risk of death at each instant $\tau \geq t$, $t \in \mathbb{R}_0^+$. The utility of an individual at each instant τ is given by

$$F(c(\tau), \alpha(\tau)) = u(c(\tau))\,\alpha(\tau)$$

where $c(\tau)$ is consumption in τ and $\alpha(\tau)$ is an indicator of vital life functions. At birth in $t_b \in [0, t]$, this indicator is given by $\alpha(t_b) = 1$. The function $u(.)$ has the usual CRRA structure. Life functions can jump to zero, i.e. the individual can die. They do so according to the following stochastic differential equation

$$d\alpha(\tau) = -dq(\tau) \tag{4}$$

where $q(\tau)$ is a Poisson process with arrival rate $\lambda(\tau, t_b)$, where $\tau - t_b$ is the age of the individual at time τ. The individual can save as described by the budget constraint

$$dk(\tau) = \{rk(\tau) + w - c(\tau)\}\,d\tau, \tag{5}$$

where $k(\tau)$ is wealth at τ, w is the constant wage and $c(\tau)$ is consumption at τ. Lifetime utility of the individual is given by

$$U(t) = \int_t^\infty e^{-\rho(\tau - t)} F(.) d\tau. \tag{6}$$

As the length of live is uncertain, the individual solves the maximization problem $\max_{\{c(\tau)\}} E_t U(t)$ subject to (4) and (5).

(a) What are the arguments of the value function for this individual?

(b) What is the value of the optimal program for this maximization problem?

(c) Formulate the Bellmann equation for this maximization problem, taking into account that the value function of a dead person is zero.

(d) Compute the first-order condition and give an economic interpretation.

(e) Compute the Keynes-Ramsey rule. What is noteworthy about this rule?