Marten Hillebrand, Hoang Van Khieu **Mathematical Methods, Part 1**
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Solution to Problem Set 5

Problem 5.1.
(i) X_t can be rewritten as

$$\begin{aligned} X_t &= a(aX_{t-2} + b\epsilon_{t-1}) + b\epsilon_t \\ &= a^2(aX_{t-3} + b\epsilon_{t-2}) + ab\epsilon_{t-1} + b\epsilon_t \\ &\dots \\ &= a^t X_0 + a^{t-1}b\epsilon_1 + a^{t-2}b\epsilon_2 + \dots + ab\epsilon_{t-1} + b\epsilon_t \\ &= a^t X_0 + \sum_{i=1}^{t} a^{t-i} b\epsilon_i \end{aligned}$$

As $|a| < 1$ and X_0 is a real number, $a^t X_0 < \infty$. Furthermore, as ϵ_i is i.i.d and normally distributed on $\mathbb{R}$, $\sum_{i=1}^{t} a^{t-i} b\epsilon_i$ is also normally distributed on $\mathbb{R}$. It follows that X_t is normally distributed on $\mathbb{R}$.
(ii) The first moment (mean) of X_t is computed as

$$\begin{aligned} \mathbb{E}[X_t] &= \mathbb{E}[a^t X_0 + \sum_{i=1}^{t} a^{t-i} b\epsilon_i] \\ &= a^t X_0 + \sum_{i=1}^{t} a^{t-i} b\mathbb{E}[\epsilon_i] \\ &= a^t X_0 + \sum_{i=1}^{t} a^{t-i} b\mu_\epsilon \\ &= a^t X_0 + b\mu_\epsilon \frac{1-a^t}{1-a} \end{aligned}$$

The second central moment (variance) of X_t is

$$\begin{aligned} \mathrm{Var}[X_t] &= \mathrm{Var}[a^t X_0 + \sum_{i=1}^{t} a^{t-i} b\epsilon_i] \\ &= \sum_{i=1}^{t} a^{2(t-i)} b^2 \mathrm{Var}[\epsilon_i] \\ &= \sum_{i=1}^{t} a^{2(t-i)} b^2 \sigma_\epsilon^2 \\ &= b^2 \sigma_\epsilon^2 \frac{1-a^{2t}}{1-a^2} \end{aligned}$$

(iii)

$$\lim_{t\to\infty} \mathbb{E}[X_t] = \lim_{t\to\infty} \left(a^t X_0 + b\mu_\epsilon \frac{1-a^t}{1-a} \right) = \frac{b\mu_\epsilon}{1-a}$$

$$\lim_{t\to\infty} \mathrm{Var}[X_t] = \lim_{t\to\infty}\left(b^2\sigma_\epsilon^2 \frac{1-a^{2t}}{1-a^2}\right) = \frac{b^2\sigma_\epsilon^2}{1-a^2}$$

If X_t is normally distributed with mean $\frac{b\mu_\epsilon}{1-a}$ and variance $\frac{b^2\sigma_\epsilon^2}{1-a^2}$, we also have that $X_{t+1} = aX_t + b\epsilon_{t+1}$ is normally distributed. Mean and variance of X_{t+1} are computed as follow.

$$\mathbb{E}[X_{t+1}] = \mathbb{E}[aX_t + b\epsilon_{t+1}] = a\frac{b\mu_\epsilon}{1-a} + b\mu_\epsilon = \frac{b\mu_\epsilon}{1-a}$$
$$\mathrm{Var}[X_{t+1}] = \mathrm{Var}[aX_t + b\epsilon_{t+1}] = a^2\frac{b^2\sigma_\epsilon^2}{1-a^2} + b^2\sigma_\epsilon^2 = \frac{b^2\sigma_\epsilon^2}{1-a^2}$$

These moments also hold for X_{t+2} and so on. So, the limits constitute an invariant normal distribution.

Problem 5.2.
(i) The consumer's decision problem reads

$$\max_{c_t^y, c_{t+1}^o, k_{t+1}} \left\{\log(c_t^y) + \beta\mathbb{E}_t[\log(c_{t+1}^o)] | c_t^y + k_{t+1} = w_t \wedge c_{t+1}^o = R_{t+1}k_{t+1} \wedge 0 \le k_{t+1} \le w_t\right\}$$

The FOCs:

$$1 = \mathbb{E}_t\Big[\frac{c_{t+1}^o}{c_t^y}\frac{1}{\beta R_{t+1}}\Big] \tag{1}$$
$$c_t^y = w_t - k_{t+1} \tag{2}$$
$$c_{t+1}^o = R_{t+1}k_{t+1} \tag{3}$$

(ii) The firm's decision problem reads

$$\max_{K_t, L_t}\left\{\theta_t K_t^\alpha L_t^{1-\alpha} - w_t L_t - R_t K_t | (K_t, L_t) \in \mathbb{R}_+^2\right\}$$

The first order conditions are

$$R_t = \alpha\theta_t K_t^{\alpha-1}L_t^{1-\alpha} = \alpha\theta_t k_t^{\alpha-1} \tag{4}$$
$$w_t = (1-\alpha)\theta_t K_t^\alpha L_t^{-\alpha} = (1-\alpha)\theta_t k_t^\alpha \tag{5}$$

where $k_t = K_t/L_t$.
(iii) Equations (1)-(5) determine the general equilibrium for all t. Note that as (3) holds for all t, it follows that $c_t^o = R_t k_t$. By substitution, one can obtain

$$k_{t+1} = \frac{\beta}{1+\beta}(1-\alpha)\theta_t k_t^\alpha \tag{6}$$

So, function $\mathcal{K}$ is given by

$$\mathcal{K}(k_t, \theta_t) = \frac{\beta}{1+\beta}(1-\alpha)\theta_t k_t^\alpha.$$

(iii) We solve for $k_{\min}$ and $k_{\max}$ using the following equations

$$\mathcal{K}(k_{\min}, \theta_{\min}) := \frac{\beta}{1+\beta}(1-\alpha)\theta_{\min}k_{\min}^\alpha = k_{\min}$$
$$\mathcal{K}(k_{\max}, \theta_{\max}) := \frac{\beta}{1+\beta}(1-\alpha)\theta_{\max}k_{\max}^\alpha = k_{\max}$$

This yields

$$k_{\min} = \Big(\frac{\beta}{1+\beta}(1-\alpha)\theta_{\min}\Big)^{1/(1-\alpha)}$$
$$k_{\max} = \Big(\frac{\beta}{1+\beta}(1-\alpha)\theta_{\max}\Big)^{1/(1-\alpha)}$$

As $\mathcal{K}(k_{\min}, \theta_{\min}) = k_{\min}$, $\mathcal{K}(k_{\max}, \theta_{\max}) = k_{\max}$, and $\mathcal{K}$ is increasing and concave in k, to verify that $\mathcal{K}(k, \theta_{\min}) \leq k \leq \mathcal{K}(k, \theta_{\max})$ for all $k \in [k_{\min}, k_{\max}]$, it suffices to show that $\mathcal{K}'(k, \theta_{\min}) < 1$ for all $k \in [k_{\min}, k_{\max}]$, where $\mathcal{K}' := \frac{\partial \mathcal{K}}{\partial k}$. For all $k \in [k_{\min}, k_{\max}]$, we have that

$$\mathcal{K}'(k, \theta_{\min}) \leq \mathcal{K}'(k_{\min}, \theta_{\min}) = \alpha < 1$$

Alternatively, one can show that

$$\lim_{k \searrow 0} \mathcal{K}'(k, \theta_{\min}) = \lim_{k \searrow 0} \quad \frac{\beta}{1+\beta}(1-\alpha)\alpha\theta_{\min} k^{\alpha-1} = \infty > 1$$

So, there is one stable set

$$[k_{\min}, k_{\max}] = \Big[\Big(\frac{\beta}{1+\beta}(1-\alpha)\theta_{\min}\Big)^{1/(1-\alpha)}, \Big(\frac{\beta}{1+\beta}(1-\alpha)\theta_{\max}\Big)^{1/(1-\alpha)}\Big].$$