Solution to Problem Set 3

Problem 3.1.

(a) First we assume that $0 < \sigma < 1$, for the case $\sigma \geq 1$, we can apply the same argument as in Problem 1.1.d. Define the budget set

$$\mathbb{B}(e^\infty, R^\infty, \bar{s}_-) = \{(c_t, s_t)_{t\in\mathbb{T}} | c_t \geq 0, c_t + s_t \leq e_t + R_t s_{t-1} \text{ for all } t \in \mathbb{T}, s_{-1} = \bar{s}_{-1} \text{ given }, \lim_{t\to\infty} q_t s_t \geq 0\},$$

where $\mathbb{T}$ is defined as in class. The lifetime utility function is written as

$$U((c_t)_{t\in\mathbb{T}}) = \sum_{t=0}^\infty \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

Iterating the budget constraint and using the NPG condition, one can obtain

$$\sum_{t=0}^\infty q_t c_t \leq \sum_{t=0}^\infty q_t e_t + R_0 \bar{s}_{-1} := M,$$

where we impose $0 < M < \infty$, and $q_t := (R_1 \cdot R_2 \cdots R_t)^{-1}$ as in class. Now define the lifetime budget set

$$\mathcal{B}(M, q, s_{-1}) := \{(c_t)_{t\in\mathbb{T}} | \sum_{t=0}^\infty q_t c_t \leq M \land c_t \geq 0 \land c_t \in \mathcal{B}(M) \land s_{-1} = \bar{s}_{-1} \text{ given}\}$$

where $q := (q_t)_{t\in\mathbb{T}}$.

So, the decision problem is written as

$$\max_{\{c_t\}_{t\in\mathbb{T}}} \left\{U((c_t)_{t\in\mathbb{T}}) | (c_t)_{t\in\mathbb{T}} \in \mathcal{B}(M)\right\}$$

(b) By the same argument in Problem 1.1.b, the Lagrangian is written as

$$\mathcal{L}((c_t)_{t\in\mathbb{T}}, \lambda) = \sum_{t=0}^\infty \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} - \lambda \left(\sum_{t=0}^\infty q_t c_t - M\right)$$

First order conditions

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 : c_t^{-\sigma} = \lambda q_t \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 : \sum_{t=0}^\infty q_t c_t = M \quad (3)$$

for all $t \in \mathbb{T}$, and $s_{-1} = \bar{s}_{-1}$ given and $\lim_{t\to\infty} q_t s_t^* = 0$ (TVC) must hold. As (2) holds for all $t$, (2) can be rewritten as

$$c_{t+1} = \beta^{1/\sigma} \left(\frac{q_t}{q_{t+1}}\right)^{1/\sigma} c_t = (\beta R_{t+1})^{1/\sigma} c_t \quad (4)$$

Using (4) to rewrite (3)

$$c_0 \sum_{t=0}^\infty (\beta q_t^{-\sigma-1})^{1/\sigma} = M \quad (5)$$
This results in
\[ c_0 = \bar{c}_0 M \quad \text{where} \quad \bar{c}_0 := \left( \sum_{t=0}^{\infty} (\beta^t q_t^{\sigma-1})^{1/\sigma} \right)^{-1}. \tag{6} \]
So for any period \( t \), we obtain
\[ c_t = \bar{c}_t M / q_t \quad \text{where} \quad \bar{c}_t := \left( \beta^t q_t^{\sigma-1} \right)^{1/\sigma} \left/ \left( \sum_{t=0}^{\infty} (\beta^t q_t^{\sigma-1})^{1/\sigma} \right) \right. \tag{7} \]
The solution to \( s_t \) is then determined using the period budget constraint.

(c) We easily see that
\[ \sum_{t=0}^{\infty} \bar{c}_t = 1 \tag{8} \]
Therefore, (7) implies that the optimal consumption expenditure \( q_t c_t \) each period is a fraction \( \bar{c}_t \) of discounted lifetime income \( M \). The consumption share \( \bar{c}_t \) is exclusively determined by consumption prices \( (q_t)_{t \in T} \) and independent of these prices if \( \sigma = 1 \).

Problem 1.2.
(i) Given arbitrary \( W_1 = e_1 + R_1 s_0 \geq -E_1 \), the decision problem is written as
\[ V_1(W_1) = \max_{c_1,s_1} \left\{ \log(c_1) + \beta \log(e_2 + s_1 R_2) \middle| c_1 \geq 0, c_1 + s_1 \leq W_1, s_1 \geq -E_1 \right\} \]
(ii) We can argue that \( c > 0 \) and that the budget constraint must bind at the optimum. So, the problem can be rewritten as
\[ V_1(W_1) = \max_{s_1} \left\{ \log(W_1 - s_1) + \beta \log(e_2 + s_1 R_2) \middle| s_1 \geq -E_1 \right\} \]
The first order condition is written as
\[ S_1(W_1) = \frac{\beta W_1 - e_2 / R_2}{1 + \beta} \tag{9} \]
\( C_1(W_1) \) is determined by \( W_1 - S_1(W_1) \). Thus
\[ C_1(W_1) = W_1 - \frac{\beta W_1 - e_2 / R_2}{1 + \beta} = \frac{W_1 + e_2 / R_2}{1 + \beta} \tag{10} \]
(iii)
\[ V_1(W_1) = \log(W_1 - S_1(W_1)) + \beta \log(e_2 + S_1(W_1) R_2) \]
\[ = \log \left( \frac{W_1 + e_2 / R_2}{1 + \beta} \right) + \beta \log \left( \frac{\beta R_2 (W_1 + e_2 / R_2)}{1 + \beta} \right) \]
\[ = (1 + \beta) u \left( \frac{W_1 + e_2 / R_2}{1 + \beta} \right) + \beta \log(\beta R_2) \]
Now we take derivative of \( V_1 \) w.r.t \( W_1 \) and obtain
\[ V'_1(W_1) = \frac{1 + \beta}{W_1 + e_2 / R_2} \tag{11} \]
We also have
\[ u'(W_1 - S_1(W_1)) = u'(C_1(W_1)) = \frac{1 + \beta}{W_1 + e_2 / R_2} \tag{12} \]
So, we conclude $V'_1(W_1) = u'(W_1 - S_1(W_1)) = u'(C_1(W_1))$.

(iv)

$$V_0(W_0) = \max_{s_0} \left\{ \log(W_0 - s_0) + \beta V_1(W_1) | s_0 \geq -E_0 \right\}$$  \hspace{1cm} (13)

(v) FOC:

$$\frac{1}{c_0} = \beta R_1 V'_1(W_1)$$  \hspace{1cm} (14)

Using (11), we obtain

$$c_0^* = \frac{W_1^* + e_2/R_2}{R_1(1 + \beta)}$$  \hspace{1cm} (15)

$$= \frac{e_1/R_1 + s_0^* + e_2/(R_1 R_2)}{\beta(1 + \beta)}$$  \hspace{1cm} (16)

$$= \frac{e_1/R_1 + (e_0 + R_0 s_{-1} - c_0^*) + e_2/(R_1 R_2)}{\beta(1 + \beta)}$$  \hspace{1cm} (17)

$$= \frac{e_0 + e_1 q_1 + e_2 q_2 + R_0 s_{-1}}{1 + \beta + \beta^2}$$  \hspace{1cm} (18)

This solution $c_0^*$ is the same as in Problem 1.1 when $\sigma = 1$ and $T = 2$. Solution $s_0^*$ can be derived as below.

$$s_0^* = e_0 + R_0 s_{-1} - c_0^*$$  \hspace{1cm} (19)

(vi)

$$c_1^* = \frac{W_1^* + e_2/R_2}{1 + \beta}$$  \hspace{1cm} (20)

$$= \frac{e_1 + R_1 s_0^* + e_2/R_2}{1 + \beta}$$  \hspace{1cm} (21)

$$= \frac{e_1 + R_1 (e_0 + R_0 s_{-1} - c_0^*) + e_2/R_2}{1 + \beta}$$  \hspace{1cm} (22)

$$= \frac{e_0 + e_1 q_1 + e_2 q_2 + R_0 s_{-1} - c_0^*}{q_1(1 + \beta)}$$  \hspace{1cm} (23)

$$= \frac{\beta}{1 + \beta + \beta^2} \frac{e_0 + e_1 q_1 + e_2 q_2 + R_0 s_{-1}}{q_1}$$  \hspace{1cm} (24)

$s_1^*$ is determined by $s_1^* = e_1 + R_1 s_0^* - c_1^*$. So, we can see that the solutions $c_1^*$ and $s_1^*$ are the same as in Problem 1.1 when $\sigma = 1$ and $T = 2$. 