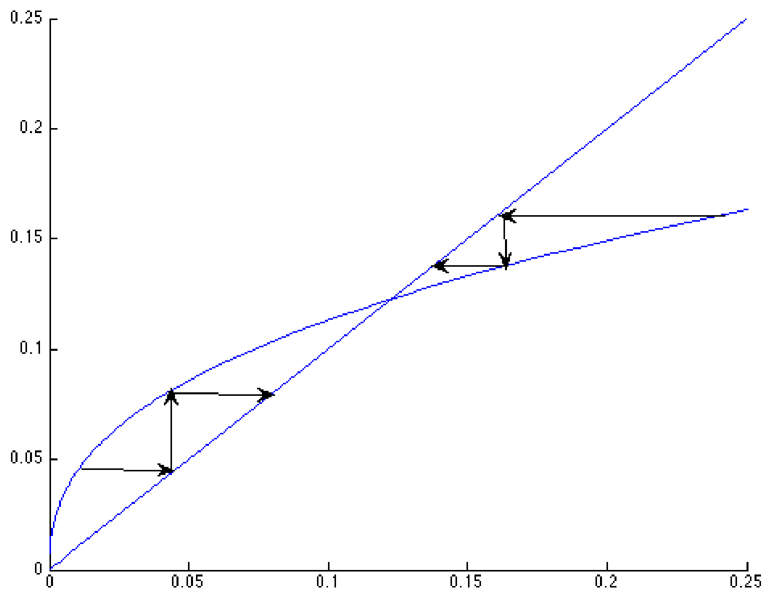
# Solution to Problem Set 2

**Problem 2.1**.
**(i.a)** $\varphi(k) = \mathcal{K}(k) = \frac{\beta}{1+\beta}(1-\alpha)Ak^{\alpha}$, $k \geq 0$. $\varphi$ is increasing and $\lim_{k \searrow 0} \varphi(k) = 0$, and $\lim_{k \to +\infty} \varphi(k) = +\infty$. Thus, $\varphi = \mathcal{K}$ maps $X = \mathbb{R}_+$ into itself.
**(i.b)** There are two fixed points $\bar{k}_1 = 0$ and $\bar{k}_2 = \left(\frac{\beta(1-\alpha)A}{1+\beta}\right)^{1/1-\alpha}$. Consider the limit $\lim_{k \searrow 0} \frac{\varphi(k)}{k} = \infty$. This implies that $\varphi(k) > k$ for $k$ close to zero. Thus, $\bar{k}_1$ is unstable. We have $|\varphi'(\bar{k}_2)| < 1$; thus, $\bar{k}_2$ is locally stable.
**(i.c)**

Figure 1: Dynamics of capital




**(ii.a)** First notice that when $1 < B < 4$, $x_1 = -\left(\frac{B-1}{3B}\right)^{1/2}$, and $x_2 = \left(\frac{B-1}{3B}\right)^{1/2}$ solve $\varphi'(x) = 0$. $\varphi$ is therefore increasing in $[-1, x_1)$, decreasing in $(x_1, x_2)$, and increasing in $(x_2, 1]$.
Consider the limits $\lim_{x \to -1} \varphi(x) = -1$, $\lim_{x \to 1} \varphi(x) = 1$, $\lim_{x \to x_1} \varphi(x) = 2B\left(\frac{B-1}{3B}\right)^{3/2}$, $\lim_{x \to x_2} \varphi(x) = -2B\left(\frac{B-1}{3B}\right)^{3/2}$.
It is straightforward to verify that $2B\left(\frac{B-1}{3B}\right)^{3/2}$ is increasing in B and less than one for $0 < B < 4$, and that $-2B\left(\frac{B-1}{3B}\right)^{3/2}$ is decreasing and greater than $-1$ for $0 < B < 4$. When $0 < B < 1$, $\varphi(x)$ is increasing, and $\lim_{x \to -1} \varphi(x) = -1$ and $\lim_{x \to 1} \varphi(x) = 1$
**(ii.b)** There are 3 fixed points $\bar{x}_1 = -1$, $\bar{x}_2 = 0$, and $\bar{x}_3 = 1$. Evaluating $\varphi'(x)$ at these fixed points to conclude that $\bar{x}_1$ and $\bar{x}_3$ are unstable; $\bar{x}_2$ is locally asymptotically stable if $0 < B < 2$ and unstable if $2 < B < 4$.

**(ii.c)**

Figure 2: $B = \frac{1}{2}$

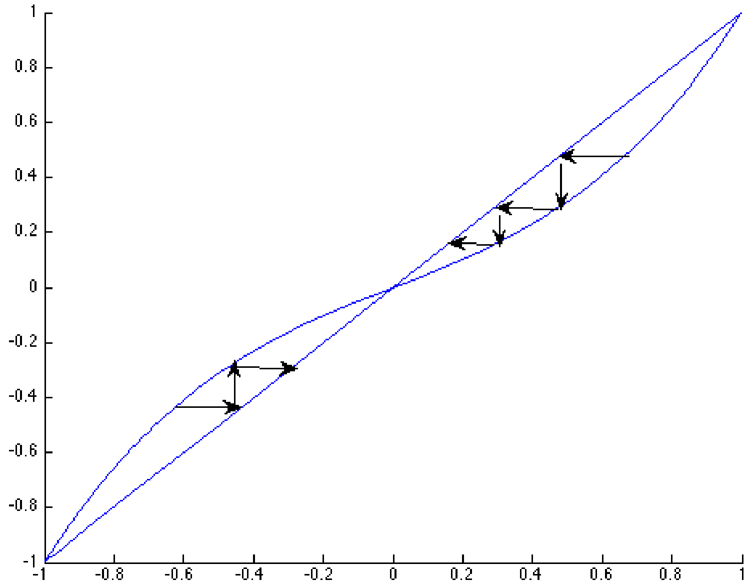



Figure 3: $B = \frac{3}{2}$

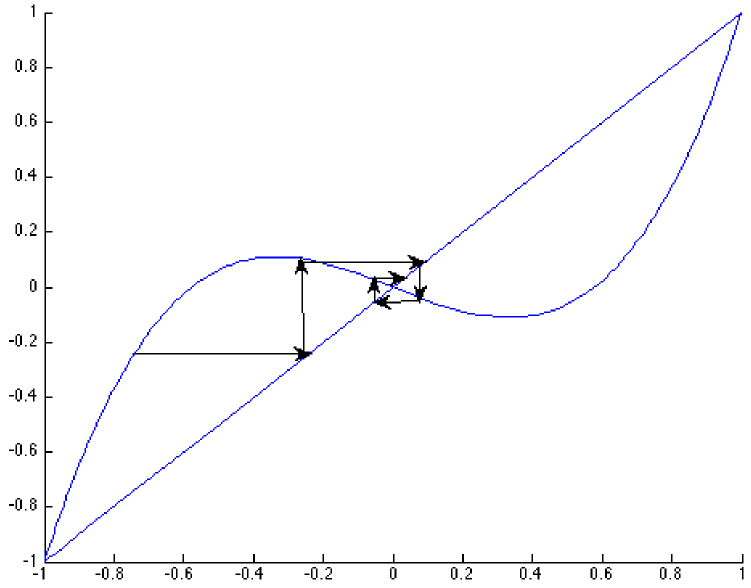
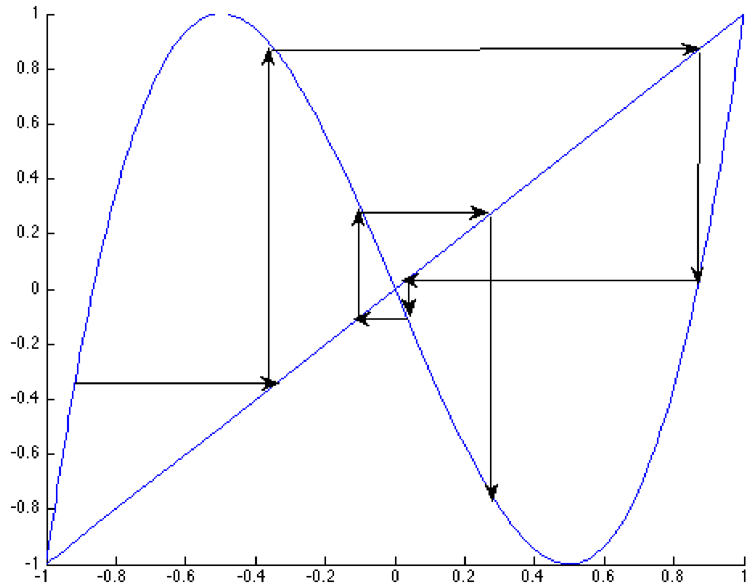


Figure 4: $B = 4$




**Problem 2.2**.
**(i)** Assume $\bar{\mathbf{x}}$ is a unique steady state, then we must have

$$\begin{aligned} \bar{\mathbf{x}} &= A\bar{\mathbf{x}} \\ \iff (I_2 - A)\bar{\mathbf{x}} &= \mathbf{0}_2 \end{aligned}$$

If $\det[I_2 - A] \neq 0$, then $\bar{\mathbf{x}} = [I_2 - A]^{-1}\mathbf{0}_2$ is a unique fixed point.

$$\begin{aligned} &\iff 1 - (a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) \neq 0 \\ &\iff \text{tr}(A) - \det(A) \neq 1 \end{aligned}$$

**(ii)**

$$\begin{aligned} &\det[(A - \lambda I_2] = 0 \\ \iff &\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0 \\ \iff &\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0 \\ \iff &\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0 \\ \iff &\lambda_{1,2} = \frac{\text{tr}(A)}{2} \pm \sqrt{\frac{(\text{tr}(A))^2}{4} - \det(A)} \end{aligned}$$

Both Eigenvalues are real if and only if $\frac{(\text{tr}(A))^2}{4} \geq \det(A)$. Otherwise, they are complex.
**(iii.a)** $\text{tr}(A_1) - \det(A_1) = 1/16 \neq 1$, so $\varphi(x)$ has a unique fixed point $\bar{x} = 0$. As $\varphi$ is linear, the Jacobian is the same as $A_1$. Eigenvalues of the Jacobian are $\lambda_{1,2} = \pm\frac{1}{4}$. So the fixed point is locally asymptotically stable.
**(iii.b)** $\text{tr}(A_2) - \det(A_2) = 3/8 \neq 1$, so $\varphi(x)$ has a unique fixed point $\bar{x} = 0$. As $\varphi$ is linear, the Jacobian is the same as $A_2$. Eigenvalues of the Jacobian are $\lambda_{1,2} = \frac{1}{4} \pm i\frac{1}{4}$. So the fixed point is locally asymptotically stable.

**(iii.c)** $\text{tr}(A_3) - \det(A_3) = 13/4 \neq 1$, so $\varphi(x)$ has a unique fixed point $\bar{x} = 0$. As $\varphi$ is linear, the Jacobian is the same as $A_3$. Eigenvalues of the Jacobian are $\lambda_1 = -1/2$, and $\lambda_2 = 5/2$. So the fixed point is unstable.
**(iv)** Let $v = [v_1 \quad v_2]'$ be the Eigenvector associated with the smaller Eigenvalue. $v$ is determined by

$$
\begin{aligned}
&[A_3 - \lambda_1 I_2]v = 0 \\
\Rightarrow &\begin{bmatrix} \frac{3}{2} & \frac{1}{4} \\ 9 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \\
\Rightarrow &v_2 = -6v_1
\end{aligned}
$$

So the stable manifold is given as

$$\mathbb{M} = \{(x, y) \in \mathbb{R}^2 | y = -6x\}.$$