Solution to Problem Set 2

Problem 2.1.
(i.a) \( \varphi(k) = K(k) = \frac{\beta}{1+\beta} (1-\alpha) A k^\alpha \), \( k \geq 0 \). \( \varphi \) is increasing and \( \lim_{k \searrow 0} \varphi(k) = 0 \), and \( \lim_{k \to +\infty} \varphi(k) = +\infty \).
Thus, \( \varphi = K \) maps \( X = \mathbb{R}_+ \) into itself.

(ii.a) First notice that when \( 1 < B < 4 \), \( x_1 = -\left( \frac{B-1}{3B} \right)^{1/2} \), and \( x_2 = \left( \frac{B-1}{3B} \right)^{1/2} \) solve \( \varphi'(x) = 0 \). \( \varphi \) is therefore increasing in \( [-1, x_1] \), decreasing in \( (x_1, x_2) \), and increasing in \( (x_2, 1] \).
Consider the limits \( \lim_{x \to -1} \varphi(x) = -1 \), \( \lim_{x \to 1} \varphi(x) = 1 \), \( \lim_{x \to x_1} \varphi(x) = 2B \left( \frac{B-1}{3B} \right)^{3/2} \), \( \lim_{x \to x_2} \varphi(x) = -2B \left( \frac{B-1}{3B} \right)^{3/2} \).
It is straightforward to verify that \( 2B \left( \frac{B-1}{3B} \right)^{3/2} \) is increasing in \( B \) and less than one for \( 0 < B < 4 \), and that \( -2B \left( \frac{B-1}{3B} \right)^{3/2} \) is decreasing and greater than \(-1\) for \( 0 < B < 4 \). When \( 0 < B < 1 \), \( \varphi(x) \) is increasing, and \( \lim_{x \to -1} \varphi(x) = -1 \) and \( \lim_{x \to 1} \varphi(x) = 1 \).

(ii.b) There are 3 fixed points \( \bar{x}_1 = -1 \), \( \bar{x}_2 = 0 \), and \( \bar{x}_3 = 1 \). Evaluating \( \varphi'(x) \) at these fixed points to conclude that \( \bar{x}_1 \) and \( \bar{x}_3 \) are unstable; \( \bar{x}_2 \) is locally asymptotically stable if \( 0 < B < 2 \) and unstable if \( 2 < B < 4 \).
(ii.c)

Figure 2: $B = \frac{1}{2}$

Figure 3: $B = \frac{3}{2}$
Problem 2.2.

(i) Assume \( \bar{x} \) is a unique steady state, then we must have

\[
\bar{x} = A\bar{x}
\]

\[\iff (I_2 - A)\bar{x} = 0 \]

If \( \det[I_2 - A] \neq 0 \), then \( \bar{x} = [I_2 - A]^{-1}0_2 \) is a unique fixed point.

\[\iff 1 - (a_{11} + a_{22}) + (a_{11} a_{22} - a_{12} a_{21}) \neq 0 \]

\[\iff \text{tr}(A) - \det(A) \neq 1 \]

(ii)

\[
\det[(A - \lambda I_2] = 0
\]

\[\iff \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0 \]

\[\iff \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11} a_{22} - a_{12} a_{21}) = 0 \]

\[\iff \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0 \]

\[\iff \lambda_{1,2} = \frac{\text{tr}(A)}{2} \pm \sqrt{\left(\frac{\text{tr}(A)}{4}\right)^2 - \det(A)} \]

Both Eigenvalues are real if and only if \( \frac{(\text{tr}(A))^2}{4} \geq \det(A) \). Otherwise, they are complex.

(iii.a) \( \text{tr}(A_1) - \det(A_1) = 1/16 \neq 1 \), so \( \varphi(x) \) has a unique fixed point \( \bar{x} = 0 \). As \( \varphi \) is linear, the Jacobian is the same as \( A_1 \). Eigenvalues of the Jacobian are \( \lambda_{1,2} = \pm \frac{1}{4} \). So the fixed point is locally asymptotically stable.

(iii.b) \( \text{tr}(A_2) - \det(A_2) = 3/8 \neq 1 \), so \( \varphi(x) \) has a unique fixed point \( \bar{x} = 0 \). As \( \varphi \) is linear, the Jacobian is the same as \( A_2 \). Eigenvalues of the Jacobian are \( \lambda_{1,2} = \frac{1}{4} \pm \frac{i}{4} \). So the fixed point is locally asymptotically stable.
(iii.c) $\text{tr}(A_3) - \det(A_3) = 13/4 \neq 1$, so $\varphi(x)$ has a unique fixed point $\bar{x} = 0$. As $\varphi$ is linear, the Jacobian is the same as $A_3$. Eigenvalues of the Jacobian are $\lambda_1 = -1/2$, and $\lambda_2 = 5/2$. So the fixed point is unstable.

(iv) Let $v = [v_1 \ v_2]'$ be the Eigenvector associated with the smaller Eigenvalue. $v$ is determined by

$$[A_3 - \lambda_1 I_2]v = 0$$

$$\Rightarrow \begin{bmatrix} 3/2 & 1/2 \\ 9 & 3/2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\Rightarrow v_2 = -6v_1$$

So the stable manifold is given as

$$\mathcal{M} = \{(x, y) \in \mathbb{R}^2 | y = -6x\}.$$