Lecture 2
The Centralized Economy: Basic features

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I Motivation

- This Lecture introduces the basic dynamic general equilibrium model of a closed economy which is at the heart of modern macroeconomics

→ *Main reference: Wickens, Chapter 2, Sections 2.1-2.4*

- **Goal**: we will analyze how to optimally allocate output between consumption and investment (ie capital accumulation) or, alternatively, between 'consumption today' and 'consumption tomorrow'

I Motivation

In this lecture we will isolate a few core aspects. Many important things will be missing. For example:

- there will be no government, no market structure (in particular: no financial markets), no money (such that all variables are in real, not in nominal terms)
- there will be no uncertainty and no sources of persistence
- the labour supply will be fixed and capital can be installed without adjustment costs
- there will be no population growth and no technical progress

I Motivation

Why do we start with such a seemingly unrealistic and simplistic macroeconomic model?

- There is a good scientific tradition to start out from simple, well-understood structures
- Complexity can always be added, but this needs to be done in a disciplined way
- Otherwise we would have to rely immediately on numerical methods which are routinely used for large-scale macroeconomic models
- But such methods will only be illuminating if the core of a model is sufficiently simple such that it can be ‘understood’
- Subsequent lectures will cover extensions and add additional features

I Motivation

The basic model of the centralized economy, notwithstanding its simplicity, has been very influential over decades

→ **Interpretations of the basic model:**

- Frank **Ramsey (1927)** introduced a similar version to study taxation issues. Hence, the model is often called the **Ramsey model**
- The model can be interpreted as a **social planning model** in which decisions are taken by the central planner, taking as given individual preferences
- The model gives rise to a **representative agent model**, in the sense that all economic agents are identical (and households and firms have the same objectives)
- Since there exists, in fact, only a single individual, the model describes a **Robinson Crusoe economy**
- The model is the basis of **neoclassical growth theory** (Solow, 1956, Cass, 1965, Koopmans, 1967)

II Basic model ingredients

Notation

- Consider a closed economy with a constant population N
- In a representative period t, we consider the following **aggregate variables** (using **capital letters**):

Y_t	output
C_t	consumption
K_t	*predetermined* level of capital available for production
I_t	gross investment undertaken within the period
S_t	savings

- Alternatively, consider these variables in **per capita form** (using **lower case letters**), ie output per capita is given by

$$y_t = \frac{Y_t}{N}$$

- Similarly:

$$c_t = \frac{C_t}{N}, \; k_t = \frac{K_t}{N}, \; i_t = \frac{I_t}{N}, \; s_t = \frac{S_t}{N}$$

II Basic model ingredients

Key equations

To capture choices between 'consumption today' and 'consumption tomorrow' in a closed economy consider 3 basic equations (**per capita form**)

1) Resource constraint (national income identity):

$$y_t = c_t + i_t, \tag{1}$$

where we use that savings are equal to investment, ie

$$s_t = y_t - c_t = i_t$$

2) Capital stock dynamics

$$\underbrace{\Delta k_{t+1}}_{k_{t+1}-k_t} = i_t - \delta k_t, \tag{2}$$

saying that Δk_{t+1}(ie net investment) results from gross investment (i_t) minus depreciation (where we assume that a constant proportion $\delta \in (0, 1)$ of the existing capital stock depreciates in period t)

II Basic model ingredients

Key equations

3) Production function

$$y_t = f(k_t) \tag{3}$$

Idea: The 'neoclassical' production function f is such that an increase in k increases output, but at a diminishing rate.

Let $k > 0$. Then:

$$f(k) > 0,\ \ f'(k) > 0, f''(k) < 0$$

Moreover:

$$\lim_{k \to 0} f'(k) \to \infty, \quad \lim_{k \to \infty} f'(k) \to 0$$

These are the so-called '*Inada-conditions*'. What do they say?

- at the origin there are infinite output gains to increasing k
- these gains decline as k becomes larger
- they eventually disappear if k becomes arbitrarily large

II Basic model ingredients

Key equations

Comment: Production function (aggregate vs. **per capita output)**

- Notice that
$$y_t = f(k_t)$$
is in per capita form
- The aggregate production function is given by
$$Y_t = F(K_t, N).$$
In neoclassical tradition, F has **constant returns to scale**, ie for any proportionate variation λ of both inputs the function F satisfies
$$F(\lambda K_t, \lambda N) = \lambda F(K_t, N) = \lambda Y_t$$
- Hence, assuming $\lambda = \frac{1}{N}$, per capita output satisfies
$$y_t = \frac{Y_t}{N} = F(k_t, 1) \equiv f(k_t)$$
- **'Notice'**: in some textbooks (eg. Wickens) you find the alternative *notation* for per capita output
$$F(k_t, 1) \equiv F(k_t)$$

II Basic model ingredients
Key equations

- We can combine eqns (1)-(3) and eliminate y_t and i_t such that the resource constraint simplifies to

$$f(k_t) = c_t + \Delta k_{t+1} + \delta k_t$$

- Since $\Delta k_{t+1} = k_{t+1} - k_t$, this equation acts like a dynamic constraint on the economy
- Equivalently, to see how this equation restricts the feasible choices of consumption over time, write it as

$$c_t = f(k_t) - k_{t+1} + (1-\delta)k_t \tag{4}$$

II Basic model ingredients

Key equations

Interpretation:

- Eqn (4), ie
$$c_t = f(k_t) - k_{t+1} + (1-\delta)k_t$$
can be read as follows:
$\longrightarrow$ Consider an initial period $t = 0$ with a given (ie *predetermined*) value k_0 (which fixes output $f(k_0)$ in period $t = 0$)
$\longrightarrow$ *Assume* there exists some rule or some regularity which tells us for the given value of k_0 how to determine the consumption level c_0.This will implicitly determine k_1.
$\longrightarrow$ If we use the same rule again in $t = 1$ we find c_1, and, implicitly, k_2
$\longrightarrow$ Continuing this recursive logic for $t = 2, 3, .., T$, we can derive the entire sequence of c and k into the infinite future (ie $T \rightarrow \infty$)
- Notice that eqn (4) is non-linear because of the term $f(k_t)$

II Basic model ingredients

Possible choices for consumption: overview

Given the just derived dynamic constraint (4), ie

$$c_t = f(k_t) - k_{t+1} + (1-\delta)k_t$$

we need some criterion or objective in order to determine optimal choices of consumption

- An extreme choice would be entirely **myopic**, ie for a given value k_0 the highest possible level of c_0 in period $t = 0$ amounts to

$$c_0^{myopic} = f(k_0) + (1-\delta)k_0$$

  Yet, this choice would imply $k_1 = 0$, ie it is **not sustainable** (in fact, it would imply zero output and zero consumption in all future periods!)

II Basic model ingredients

Possible choices for consumption: overview

- A more reasonable criterion is to impose that **consumption** levels should be **sustainable**, ie consumption should be maximized in each period
- We will consider two alternatives: the so-called **golden rule solution** and an **optimal solution**
- The key difference between the two solution concepts is that under the optimal solution future consumption will be **discounted**, while the golden rule ignores discounting

III Golden rule solution

- The golden rule solution is derived from a long-term objective:
  $\longrightarrow$ it maximizes the (constant) amount of per capita consumption in each period
  $\longrightarrow$ by doing so, it treats members of different generation alike ('golden rule')
- Hence, going back to eqn (4), ie

$$c_t = f(k_t) - k_{t+1} + (1-\delta)k_t$$

  we consider a long-run (or 'steady-state') situation in which all per capita variables are constant (in particular $k_t = k_{t+1} = k$), leading to

$$c = f(k) - \delta k \tag{5}$$

- Eqn (5) implies that net investment will be zero, ie
  the only investment undertaken is such that it replaces depreciated capital, facilitating a constant capital stock over time

III Golden rule solution

- Given the steady-state resource constraint (5), ie

$$c = f(k) - \delta k,$$

how should one optimally choose c ?

→ To find the golden rule solution we solve the maximization problem

$$\max_{k} \; f(k) - \delta k$$

- The golden rule capital stock k_{GR} is implicitly characterized by the first-order condition

$$\frac{dc}{dk} = f'(k_{GR}) - \delta = 0,$$

and the second-order condition, evaluated at k_{GR},

$$\frac{d^2c}{dk^2} = f''(k_{GR}) < 0$$

ensures that k_{GR} is a maximum (and not a minimum)

III Golden rule solution

Uniqueness:
Given the assumptions on f, the optimum k_{GR} which solves

$$f'(k_{GR}) = \delta \tag{6}$$

is unique and the associated unique consumption level c_{GR} is given by

$$c_{GR} = f(k_{GR}) - \delta k_{GR} \tag{7}$$

Interpretation of the golden rule solution:

- Eqn (6) says that steady-state per capita consumption will be maximized if the marginal product of k equals the depreciation rate δ
- Below the level k_{GR} a marginal increase in k increases c, since the marginal gain in output (ie $f'(k)$) exceeds the output cost of replacing depreciated capital
- Above the level k_{GR} a marginal increase in k would decrease c, since the marginal gain in output (ie $f'(k)$) is smaller than the output cost of replacing depreciated capital

III Golden rule solution

Comparative statics

Let us use the golden rule solution to introduce the notion of **comparative statics:**

- Idea: how do long-run (steady-state) solutions of endogenous variables change if an exogenous parameter changes?
- Typically we can sign these changes, by using the information embodied in the functional forms that are used

Particular example:

$\longrightarrow$ Assume the rate δ at which capital depreciates increases...

$\longrightarrow$ ...How do k_{GR} and c_{GR} react to the exogenous change in δ ?

III Golden rule solution

Comparative statics

Particular comparative statics example: increase in δ

- Recall that the first-order optimality condition

$$f'(k_{GR}) = \delta$$

establishes only an **implicit** dependence of k_{GR} on δ, ie we cannot directly differentiate k_{GR} with respect to δ

- Yet, since this optimality condition will be satisfied for any exogenous value δ, we can write it as an identity

$$f'(k_{GR}(\delta)) - \delta \equiv 0 \tag{8}$$

- Differentiating (8) w.r.t. δ (where we use the chain rule) yields

$$f''(k_{GR}) \cdot \frac{dk_{GR}}{d\delta} - 1 \equiv 0,$$

implying

$$\frac{dk_{GR}}{d\delta} = \frac{1}{f''(k_{GR})} < 0 \tag{9}$$

$\longrightarrow$ an increase in δ makes the accumulation of capital more costly, leading to a decline in k_{GR}

III Golden rule solution
Comparative statics

What about the reaction of c_{GR} to a change in δ?

- To respect the implicit dependence of k_{GR} on δ, express (7) as

$$c_{GR} = f(k_{GR}(\delta)) - \delta k_{GR}(\delta)$$

- Differentiating c_{GR} with respect to δ gives:

$$\begin{aligned} \frac{dc_{GR}}{d\delta} &= \frac{d[f(k_{GR}(\delta)) - \delta k_{GR}(\delta)]}{d\delta} \\ &= \underbrace{[f'(k_{GR}) - \delta]}_{=0} \frac{dk_{GR}}{d\delta} - k_{GR}(\delta) \\ &= -k_{GR}(\delta) < 0 \end{aligned}$$

$\rightarrow$ an increase in δ leads also to a decline in c_{GR}

III Golden rule solution

Comparative statics

Comment:

- To derive comparative statics results from implicit relationships like

$$f'(k_{GR}) = \delta$$

  there exist alternative techniques

- In particular, if one **totally differentiates** the relationship at the equilibrium one obtains

$$f''(k_{GR}) \cdot dk = d\delta,$$

  which can be rearranged to confirm (9), ie

$$\frac{dk_{GR}}{d\delta} = \frac{1}{f''(k_{GR})} < 0.$$

III Golden rule solution

What is missing?

- Lecture 1 argued that modern macroeconomics attempts to base the analysis on micro-founded welfare criteria, consistent with optimizing behaviour of the representative consumer
- The golden rule analysis carefully incorporates the dynamic constraint relating to capital stock dynamics...
- ...but it is silent on whether there exists an **individual welfare measure** that would generate the golden rule solution

III Golden rule solution

What is missing?

- In particular, the golden rule analysis pretends that individuals value consumption today and consumption tomorrow in the same way
- But this is not a satisfactory assumption, given the observed **impatience** in decisions of consumers
- This aspect is captured by the so-called **optimal solution** (meaning that the optimality criterion corresponds to a micro-founded welfare objective which incorporates impatience)

IV Optimal solution

Objective

- Let the representative period be denoted by t
- Assume there exists in the initial period $t = 0$ a predetermined per capita capital stock k_0
- Let V_0 denote the present value of current and future utility, as given by:

$$V_0 = \sum_{t=0}^{\infty} \beta^t U(c_t), \tag{10}$$

  where the instantaneous utility $U_t = U(c_t)$ satisfies $U'(c_t) > 0$ and $U''(c_t) < 0$, ie within any period additional consumption increases utility but at a diminishing rate
- The objective V_0 is additively separable which makes it easy to compare utility between periods
- Future utility is discounted by the constant factor β which satisfies $0 < \beta < 1$
- Alternatively, we can define the corresponding discount rate $\theta > 0$, with:

$$\beta = \frac{1}{1+\theta}$$

IV Optimal solution

Objective

- The goal pursued by the optimal solution is to choose current and future consumption such that the objective (10), ie

$$V_0 = \sum_{t=0}^{\infty} \beta^t U(c_t),$$

will be maximized subject to the above established dynamic constraint (4), ie

$$c_t = f(k_t) - k_{t+1} + (1-\delta)k_t$$

- We will solve this dynamic optimization problem by using the Lagrange multiplier technique

IV Optimal solution

Solution based on Lagrange multipliers

→ Consider the objective $\mathcal{L}$ which incorporates the resource constraint (4)
→ In order to maximize (10) s.t. (4) we optimize, equivalently,

$$\mathcal{L} = \sum_{t=0}^{\infty} \{\beta^t U(c_t) + \lambda_t [f(k_t) - c_t - k_{t+1} + (1-\delta)k_t]\}$$

over the choice variables $\{c_t,\ k_{t+1},$ and $\lambda_t;\ \forall t \geqslant 0\}$
→ λ_t is the Lagrange multiplier t periods ahead, measuring the shadow value of an additional unit of period t income (in terms of utility of period 0)

First-order optimality conditions ('FOCs', interior) w.r.t. c_t, k_t, λ_t :

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t U'(c_t) - \lambda_t = 0 \qquad t \geqslant 0 \qquad (11)$$

$$\frac{\partial \mathcal{L}}{\partial k_t} = \lambda_t [f'(k_t) + (1-\delta)] - \lambda_{t-1} = 0 \qquad t > 0 \qquad (12)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = f(k_t) - c_t - k_{t+1} + (1-\delta)k_t = 0 \qquad t \geqslant 0 \qquad (13)$$

Transversality condition: $$\lim_{t\to\infty} \underbrace{\beta^t \cdot U'(c_t)}_{\lambda_t} \cdot k_{t+1} = 0 \qquad (14)$$

IV Optimal solution

Solution based on Lagrange multipliers

Comment: How to read the just derived equations (11)-(14)?

- These are **necessary conditions for optimality**
- The sufficient conditions for a maximum are satisfied, given our assumptions on functional forms
- *Notice:* The concept of intertemporal optimality applies to sequences of variables, ie the equations form a **system of difference equations** characterizing the behaviour of the equilibrium over time
- Crucial for the exact time paths of variables consistent with such system: **initial** and **terminal** conditions

IV Optimal solution

Solution based on Lagrange multipliers

Comment: Initial condition

- By assumption, the economy starts to operate in $t = 0$, taken as given the predetermined level of the per capita capital stock k_0
  $\longrightarrow$ k is the single **predetermined (state) variable** of the system
- In period $t = 0$, the per capita consumption level c_0 can be freely chosen
  $\longrightarrow$ c is the single **forwardlooking (control) variable** w/o initial condition
- These features will become relevant when we discuss stability issues below

IV Optimal solution

Solution based on Lagrange multipliers

Comment: Terminal condition

- The **transversality condition** (14), ie

$$\lim_{t\to\infty} \underbrace{\beta^t \cdot U'(c_t)}_{\lambda_t} \cdot k_{t+1} = 0,$$

  closes the system by backward induction from the (distant) future

- To see how this can be made operational, consider first some large and finite value of t, ie a distant period somewhere far out in the future...

IV Optimal solution

Solution based on Lagrange multipliers

Comment: Terminal condition

- ...For any finite value of t, the term

$$\beta^t \cdot U'(c_t) \cdot k_{t+1} = \lambda_t \cdot k_{t+1}$$

describes the present value of the utility that could be obtained if k_{t+1} (ie the capital stock for the next period resulting from investment decisions in t) will be consumed at t rather than being left for production for $t+1$

- If this particular value of t marks the terminal period it cannot be optimal, not to consume everything in the terminal period
- **Infinite horizon analogy**: There exists no terminal period, but as $t \longrightarrow \infty$, it cannot be optimal to postpone consumption forever, ie

$$\lim_{t \to \infty} \lambda_t k_{t+1} = 0,$$

as specified by (14).

IV Optimal solution

Solution based on Lagrange multipliers

Simplification of the FOCs:

- Let us reconsider the FOCs (11) and (12), ie

$$\begin{aligned} \beta^t U'(c_t) - \lambda_t &= 0 \qquad t \geqslant 0 \\ \lambda_t[f'(k_t) + (1-\delta)] - \lambda_{t-1} &= 0 \qquad t > 0 \end{aligned}$$

- We can obtain the Lagrange multiplier from the first eqn and substitute for λ_t and λ_{t-1}, respectively, in the second eqn, leading to

$$\beta^t U'(c_t)[f'(k_t) + (1-\delta)] = \beta^{t-1} U'(c_{t-1}) \qquad t > 0$$

- Equivalently, after dividing by β^{t-1} and updating of all terms by one period, we can rewrite this eqn as

$$\beta U'(c_{t+1})[f'(k_{t+1}) + (1-\delta)] = U'(c_t), \qquad t \geqslant 0 \qquad (15)$$

which is the so-called **consumption Euler equation**

IV Optimal solution

What do we get?

2 key equations:

- Recall from above that via eqn (13) the optimization preserved the dynamic resource constraint (4)
- In sum, the (consolidated) intertemporal equilibrium consists of the **consumption Euler equation** and the **resource constraint**, ie we have $\forall t \geqslant 0$:

$$U'(c_t) = \beta U'(c_{t+1})[f'(k_{t+1}) + (1-\delta)] \qquad (16)$$

$$c_t = f(k_t) - k_{t+1} + (1-\delta)k_t \qquad (17)$$

$\longrightarrow$ We have reduced the dynamics to a non-linear two-dimensional dynamic system in c and k with one initial condition (k_0) and one terminal condition, as given by the transversality condition (14)

- Before we analyze the system (16)-(17), we will give some more interpretation to the consumption Euler equation

IV Optimal solution

What do we get?

Interpretation of the consumption Euler equation:

- The **consumption Euler equation (16),** ie

$$\beta U'(c_{t+1})[f'(k_{t+1}) + (1-\delta)] = U'(c_t), \qquad t \geqslant 0$$

  is the fundamental dynamic equation in intertemporal optimization problems in which consumers actively decide about how to choose between 'consumption today' and 'consumption tomorrow'

- In eqn (16), 'today' corresponds to $t = 0$. Since the optimization holds $\forall t \geqslant 0$, the recursive nature of the FOCs implies that 'tomorrow' covers not only $t = 1$, but all subsequent future periods, ie $t = 2, 3, \ldots T \ldots$etc.

IV Optimal solution

What do we get?

- The **consumption Euler equation**

$$\beta U'(c_{t+1})[f'(k_{t+1}) + (1-\delta)] = U'(c_t)$$

can be seen as an **intertemporal arbitrage condition**, saying that at the optimum the representative consumer must be indifferent between consuming a marginal unit of c, yielding extra utility

$$U'(c_t),$$

or, alternatively, investing this unit and consuming the return one period later, yielding extra utility

$$\beta U'(c_{t+1})[f'(k_{t+1}) + (1-\delta)]$$

- The discount factor β ensures that consumption today and tomorrow will be comparable in terms of utility

IV Optimal solution

Long-run (steady-state) features of the optimal solution

- Let us go back to the pair of equilibrium eqns (16) and (17), ie

$$\begin{aligned} U'(c_t) &= \beta U'(c_{t+1})[f'(k_{t+1}) + (1-\delta)] \\ c_t &= f(k_t) - k_{t+1} + (1-\delta)k_t \end{aligned}$$

- **Long-run ('steady-state') equilibria** exhibit constant variables
- From (16), the optimal long-run (per capita) levels k^* and c^* satisfy

$$U'(c^*) = \beta U'(c^*)[f'(k^*) + (1-\delta)],$$

  implying

$$f'(k^*) = \frac{1}{\beta} - 1 + \delta = \delta + \theta \tag{18}$$

- From (17):

$$c^* = f(k^*) - \delta k^* \tag{19}$$

IV Optimal solution

Long-run (steady-state) features of the optimal solution

→ **Steady states of the optimal solution** satisfy (18) and (19), ie

$$\begin{aligned} f'(k^*) &= \delta + \theta \\ c^* &= f(k^*) - \delta k^* \end{aligned}$$

- Eqns (18) and (19) can be solved sequentially for k^* and c^*
- Given the assumptions on f, there exists a **unique steady state**

Interpretation of the (steady-state) optimal solution:

- The optimal solution has $k^* < k_{GR}$, since $\delta + \theta > \delta$
- Moreover, $c^* < c_{GR}$, since c^* does *not* maximize $f(k) - \delta k$
- These findings reflect the role of θ : because of impatience the representative consumer does not reach the higher long-run consumption level c_{GR}

IV Optimal solution
Dynamics of the optimal solution

→ **Stability of the steady state?**

- Recall from above that **dynamics** are governed by eqns (16) and (17), ie

$$\begin{aligned} U'(c_t) &= \beta U'(c_{t+1})[f'(k_{t+1}) + (1-\delta)] \\ c_t &= f(k_t) - k_{t+1} + (1-\delta)k_t, \end{aligned}$$

  ie a non-linear two-dimensional dynamic system in c and k with one initial condition (k_0) and one terminal condition (ie the TV-condition (14))
- It can be shown that the dynamics are stable in a particular sense, ie the system is (locally) **saddlepath-stable**

IV Optimal solution

Dynamics of the optimal solution

Saddlepath-stability:

- **Saddlepath-stability** means that for any value k_0 close to the long-run value k^* there exists a unique value c_0 which
  i) satisfies all optimality conditions and
  ii) sets in motion sequences $\{c_t, k_{t+1}\}_{t=0}^{t=\infty}$ that ultimately converge against the long-run values c^* and k^*
- To calculate analytically the saddlepath requires some knowledge of matrix algebra...
- ...but the saddlepath-stable behaviour can be illustrated with a **phase diagram** which summarizes the dynamic forces of a linearized version of eqns (16) and (17)

IV Optimal solution

Dynamics of the optimal solution

- Consider eqn (16), which displays **non-linear dynamics** in c, ie

$$U'(c_t) = \beta U'(c_{t+1})[f'(k_{t+1}) + (1-\delta)]$$

- To obtain **locally linear dynamics** in c, approximate $U'(c_{t+1})$, using a first-order Taylor expansion around the point c_t, such that

$$U'(c_{t+1}) \simeq U'(c_t) + U''(c_t) \cdot \underbrace{\Delta c_{t+1}}_{c_{t+1}-c_t} \iff \frac{U'(c_{t+1})}{U'(c_t)} \simeq 1 + \frac{U''(c_t)}{U'(c_t)} \cdot \Delta c_{t+1}$$

- Use this approximation in eqn (16) to get

$$\begin{aligned} 1 + \frac{U''(c_t)}{U'(c_t)} \cdot \Delta c_{t+1} &= \frac{1}{\beta[f'(k_{t+1}) + (1-\delta)]} \\ \Delta c_{t+1} &= \underbrace{-\frac{U'(c_t)}{U''(c_t)}}_{>0} \cdot \left[1 - \frac{1}{\beta[f'(k_{t+1}) + (1-\delta)]}\right] \end{aligned} \quad (20)$$

IV Optimal solution

Dynamics of the optimal solution

- The **phase diagram** will be organized around eqns (17) and (20), ie

$$\begin{aligned}\Delta k_{t+1} &= f(k_t) - \delta k_t - c_t \\ \Delta c_{t+1} &= -\frac{U'(c_t)}{U''(c_t)} \cdot \left[1 - \frac{1}{\beta[f'(k_{t+1}) + (1-\delta)]}\right]\end{aligned}$$

- Notice that if $c_t = c^*$ and $k_t = k^*$ then $\Delta k_{t+1} = \Delta c_{t+1} = 0$
- **Dynamic implication** of **eqn (17):** it features no dynamics in c, only in k such that

$$\Delta k_{t+1} \lesseqqgtr 0 \text{ if } c_t \gtreqqless f(k_t) - \delta k_t$$

- **Dynamic implication** of **eqn (20):** it features no dynamics in k, only in c such that

$$\Delta c_{t+1} \lesseqqgtr 0 \text{ if } k_{t+1} \gtreqqless k^*,$$

- These informations can be combined to represent the dynamics in c_t and k_t via a phase diagram

IV Optimal solution

Dynamics of the optimal solution

Comments on the phase diagram of the linearized dynamics in c_t and k_t

- Arrows indicate regions of stability and instability around $k^* > 0$, $c^* > 0$
- For any initial departure of the state variable such that $k_0 \neq k^*$:
  Saddlepath configuration, i.e. there exists a unique choice of the control variable c_0 such that the economy 'jumps' on the saddlepath and converges over time towards the steady state k^*, c^*
- How does consumption optimally react along the saddlepath?
  i) Consider a **temporary negative shock** to the **capital stock**: $k_0 < k^*$.
  → The saddlepath is such that on impact $c_0 < c^*$ will be optimal
  → Thus, **temporarily**, consumption will be smaller than c^* such that some output can be diverted to rebuild the capital stock
  → This flexible short-run response of consumption is optimal, since it ensures that the long-run level c^* remains feasible
  ii) Consider a **temporary positive shock** to the **capital stock:** $k_0 > k^*$.
  → The reverse response pattern will be optimal, ie $c_0 > c^*$
  → Temporarily, consumption can be larger than c^*,
  w/o endangering the feasibility of the long-run level c^*

IV Optimal solution

Dynamics of the optimal solution

Comments on the phase diagram of the linearized dynamics in c_t and k_t

- Important information not yet used: (i) $k \geqslant 0$, and (ii) TV-condition (14) $\rightarrow$ For all other choices of c_0 (ie off the saddlepath), the dynamics ultimately drift away from k^*, c^*
- Such choices can be ruled out because the economy would eventually hit either: a **'path of rising consumption and falling capital'** on which k would become negative (but this cannot be)
  or: a **'path of falling consumption and rising capital'** on which the present value of lifetime consumption would become smaller than the present value of lifetime income (but this cannot be optimal)
- In sum, saddlepath-stability implies that the system is not only stable, but that the dynamics towards the steady state are uniquely determined

Annex: Interpreting the Euler equation in a 2-period set-up

Let us look once more at the **consumption Euler equation:**

$$\beta U'(c_{t+1})[f'(k_{t+1}) + (1-\delta)] = U'(c_t), \qquad t \geqslant 0$$

- The intertemporal optimality condition captured by the Euler equation needs to be satisfied between any two periods, $\forall t \geqslant 0$
- The reasoning can be llustrated by considering a simple **two-period analysis**
- The idea is to interpret the consumption Euler equation as an optimality condition which ensures that in equilibrium the **marginal rate of substitution along an indifference curve** and the **marginal rate of transformation along the intertemporal production possibility frontier (IPPF)** will be identical between any two periods

Annex: Interpreting the Euler equation in a 2-period set-up

Utility (2-period representation):

- Consider
$$\widetilde{V}_t = U(c_t) + \beta U(c_{t+1}),$$
ie we impose that consumption beyond period $t+1$ will not be affected
- To characterize implicitly combinations of c_t and c_{t+1} which leave utility $\widetilde{V}_t$ constant, we consider the total differential of $\widetilde{V}_t$ (with: $d\widetilde{V}_t = 0$)
$$0 = d\widetilde{V}_t = dU_t + \beta dU_{t+1} = U'(c_t)dc_t + \beta U'(c_{t+1})dc_{t+1},$$
leading to
$$\frac{dc_{t+1}}{dc_t} = -\frac{U'(c_t)}{\beta U'(c_{t+1})} < 0 \tag{21}$$

Annex: Interpreting the Euler equation in a 2-period set-up

- The expression (21), ie

$$\frac{dc_{t+1}}{dc_t} = -\frac{U'(c_t)}{\beta U'(c_{t+1})} < 0$$

  measures the (negative) slope of indifference curves of $\widetilde{V}_t$ in $c_t - c_{t+1}$ −space

- Since

$$\frac{d^2 c_{t+1}}{dc_t^2} = -\frac{U''(c_t)}{\beta U'(c_{t+1})} > 0$$

  indifference curves become flatter as c_t increases

Annex: Interpreting the Euler equation in a 2-period set-up

Resource constraint (2-period representation):

- Consider

$$c_t = f(k_t) - k_{t+1} + (1-\delta)k_t \;\; and \;\; c_{t+1} = f(k_{t+1}) - k_{t+2} + (1-\delta)k_{t+1}$$

- When totally differentiating these 2 equations we impose
  $dk_t = 0$ (since k_t is predetermined) and
  $dk_{t+2} = 0$ (since we want the capital stock beyond $t+2$ to be unchanged),
  yielding

$$dc_t = -dk_{t+1} \;\; \text{and} \;\; dc_{t+1} = f'(k_{t+1})dk_{t+1} + (1-\delta)dk_{t+1},$$

  or, by combining these two expressions (via elimination of dk_{t+1})

$$\frac{dc_{t+1}}{dc_t} = -[f'(k_{t+1}) + (1-\delta)] < 0 \tag{22}$$

Annex: Interpreting the Euler equation in a 2-period set-up

- The expression (22), ie

$$\frac{dc_{t+1}}{dc_t} = -[f'(k_{t+1}) + (1-\delta)] < 0$$

  measures the (negative) slope of the intertemporal production possibility frontier (IPPF) in $c_t - c_{t+1}$−space

- Using $k_{t+1} = f(k_t) - c_t + (1-\delta)k_t$, we get

$$\frac{d^2 c_{t+1}}{dc_t^2} = f''(k_{t+1}) < 0,$$

  ie the IPPF becomes steeper as c_t increases

Annex: Interpreting the Euler equation in a 2-period set-up

Integrated 2-period representation:

- Combining (21) and (22), ie

$$\frac{dc_{t+1}}{dc_t} = -\frac{U'(c_t)}{\beta U'(c_{t+1})} < 0$$

  and

$$\frac{dc_{t+1}}{dc_t} = -[f'(k_{t+1}) + (1-\delta)] < 0$$

  gives the consumption Euler equation

$$\beta U'(c_{t+1})[f'(k_{t+1}) + (1-\delta)] = U'(c_t)$$

- Thus, the consumption Euler equation captures a point of optimality where the IPPF is tangent to an indifference curve