Part I
Modelling Money in General Equilibrium: a Primer
Lecture 4
The Basic MIU model: Value function solution

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I Motivation

- This lecture goes back to the observation that Walsh (chapter 2) solves the basic MIU model with the **value function approach,** while we used in Lecture 2, alternatively, the **Lagrange multiplier approach**

- Against this background, the goal of this lecture is threefold, ie we will
  1) give a brief introduction to dynamic programming and the concept of a value function
  (For details, see: L. Ljungqvist and T. Sargent, *Recursive Macroeconomic Theory*, Chapter 3, MIT Press, 2nd edition, 2004)
  2) consider a simple and fully tractable example economy and compare the two solution approaches
  3) confirm the optimality conditions established in chapter 2 by Walsh

II Value function approach

Basics

- Assume we want to find an infinite sequence of a control variable $\{c_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad \beta \in (0,1), \tag{1}$$

subject to the dynamic constraint

$$\omega_{t+1} = g(\omega_t, c_t), \tag{2}$$

where ω denotes a state variable with ω_0 given

- Let the **value function** $V(\omega)$ express the **optimal** value of the above problem for any feasible initial value. In particular, define

$$V(\omega_0) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where the maximization is subject to $\omega_{t+1} = g(\omega_t, c_t)$ and ω_0 given

II Value function approach
Basics

- Under the **Lagrange multiplier approach** we solved directly for the infinite sequence $\{c_t\}_{t=0}^{\infty}$
- Alternatively, **dynamic programming** seeks to find a time-invariant **policy function** h which maps the state ω_t into the control variable c_t. The optimal sequence $\{c_t\}_{t=0}^{\infty}$ will be indirectly generated by a repeated application of the two functions

$$\begin{aligned} c_t &= h(\omega_t) \\ \omega_{t+1} &= g(\omega_t, c_t), \end{aligned}$$

ie the policy function and the dynamic constraint, with ω_0 given

II Value function approach

Basics

- *Assume we knew* $V(\omega)$. Of course, we cannot expect to know $V(\omega)$, since we have not yet solved the problem, but let us proceed on faith
- If we knew $V(\omega)$ then the policy function h could be computed by solving for each feasible value of ω the problem

$$\max_{c} \{u(c) + \beta V(\widetilde{\omega})\} \quad s.t. \;\; \widetilde{\omega} = g(\omega, c), \;\; \text{and } \omega \text{ given,} \tag{3}$$

  exploiting the recursive nature of the original maximization problem (and where $\widetilde{\omega}$ denotes the value of ω in the next period)
- But we don't know yet the value function $V(\omega)$!
- In other words, rather than to find the infinite sequence $\{c_t\}_{t=0}^{\infty}$ we have transformed the problem such that we need to find the **value function** $V(\omega)$ and the **policy function** $c = h(\omega)$ that solve the maximization problem

II Value function approach

Basics

- The task is to solve jointly for $V(\omega)$ and $h(\omega)$ which are linked by the **Bellman equation**

$$V(\omega) = \max_{c} \{u(c) + \beta V[g(\omega, c)]\} \tag{4}$$

- The maximizer of the RHS of eqn (4) is a policy function $c = h(\omega)$ that satisfies

$$V(\omega) = u(h(\omega)) + \beta V[g(\omega, h(\omega))] \tag{5}$$

- Notice that (4) or (5) are **functional equations to be solved for the pair of unknown functions** $V(\omega)$ and $h(\omega)$

II Value function approach

Features of the solution

- There exist various methods for solving the Bellman equation, depending on the precise nature of the functions u and g
- Under certain assumptions - like the concavity of $u(c)$ and the convexity and compactness of the set $\{(\omega_{t+1}, \omega_t) : \omega_{t+1} \leq g(\omega_t, c_t),\ c_t \in R\}$ - it turns out that the solution exhibits the following elements:

1) The functional equation (4) has a unique strictly concave solution $V(\omega)$

2) This solution is approached in the limit as $j \longrightarrow \infty$ by iterations on

$$V_{j+1}(\omega) = \max_{c}\{u(c) + \beta V_j(\widetilde{\omega})\} \ \ s.t. \ \ \widetilde{\omega} = g(\omega, c), \text{ and } \omega \text{ given},$$

starting from an initial functional guess $V_0(\widetilde{\omega})$.
This convergence result leads to a solution procedure which is called **value function iteration**

II Value function approach

Features of the solution

3) There exists a unique and time invariant optimal policy of the form $c_t = h(\omega_t)$. The derivation of the policy function uses the optimality condition

$$u'(c) + \beta \frac{\partial g(\omega, c)}{\partial c} V'[g(\omega, c)] = 0, \tag{6}$$

resulting from the maximization of the RHS of (4) w.r.t. c

4) The value function $V(\omega)$ is implicitly characterized by

$$\begin{aligned} V'(\omega) \quad = \quad & \beta \frac{\partial g(\omega, h(\omega))}{\partial \omega} V'[g(\omega, h(\omega))] + \\ & \underbrace{\left\{u'(h(\omega)) + \beta \frac{\partial g(\omega, h(\omega))}{\partial h} V'[g(\omega, h(\omega))]\right\}}_{=0} \frac{\partial h(\omega)}{\partial \omega}, \end{aligned}$$

resulting from the maximization of the RHS of (5) w.r.t. ω.

Using (6), this simplifies via the envelope theorem to the expression

$$V'(\omega) = \beta \frac{\partial g(\omega, h(\omega))}{\partial \omega} V'[g(\omega, h(\omega))] \tag{7}$$

II Value function approach

Comment

- These concepts may seem rather abstract, but they are not when used in practice
- To see the intuition behind them we will work through a particular example
- For this example we will verify that the two solution approaches (Value function approach and Lagrange technique) lead to the same results

III Example: Value function vs. Lagrange approach

- As a particular example of equations (1) and (2), consider the neoclassical growth framework with logarithmic preferences

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t) \quad \beta \in (0,1),$$

and Cobb-Douglas production function within the dynamic constraint

$$k_{t+1} = g(k_t, c_t) = Ak_t^{\alpha} - c_t, \quad A > 0,\ \alpha \in (0,1),$$

where k denotes the single state variable with k_0 given

- **Comment:**
$\longrightarrow$ For this example, the timing is chosen to be in line with (1) and (2) such that k simply replaces ω
$\longrightarrow$ This implies that from the perspective of period t the variable k_t is predetermined (rather than k_{t-1}, as assumed by Walsh in chapter 2)
$\longrightarrow$ For simplicity, we assume $\delta = 1$ (ie full depreciation of capital)

III Example: Value function vs. Lagrange approach

Solution via value function iteration

- Let us use
$$\widetilde{k} = g(k, c) = Ak^{\alpha} - c$$
- The Bellman equation (5) for the example at hand becomes:
$$\begin{aligned} V(k) &= \max_{c} \{u(c) + \beta V[g(k, c)]\} \quad \text{and } k \text{ given} \\ &= \max_{c} \{\ln(c) + \beta V[Ak^{\alpha} - c]\} \quad \text{and } k \text{ given} \end{aligned} \tag{8}$$
- The task is to solve jointly for the value function $V(k)$ and the policy function $c = h(k)$ which satisfy
$$\begin{aligned} V(k) &= u(h(k)) + \beta V[g(k, h(k))] \\ &= \ln(h(k)) + \beta V[Ak^{\alpha} - h(k)] \end{aligned} \tag{9}$$

III Example: Value function vs. Lagrange approach

Solution via value function iteration

To find the pair of functions $V(k)$ and $h(k)$ we employ the procedure of **value function iteration:**

$\longrightarrow$ for a given initial guess about the value function, called $V_0(k)$, we optimize the RHS of (8) over c, establish thereby a policy function $h_1(k)$ and insert it into the RHS of (8) to obtain a new value function $V_1(k)$

$\longrightarrow$ given this new function $V_1(k)$, we optimize, again, over c, to obtain a new policy function $h_2(k)$ and a new value function $V_2(k)$

$\longrightarrow$ we iterate on this procedure until convergence has been achieved, ie until we have found functions $h_\infty(k) = h(k)$ and $V_\infty(k) = V(k)$ which satisfy (9)

III Example: Value function vs. Lagrange approach

Solution via value function iteration

Value function iteration: **Initial functional guess** ($j = 0$)

- Assume
$$V_0(k) = 0$$
This guess holds for all feasible values of k, including $\widetilde{k}$
- Max of RHS of (8) over c yields (trivially!)
$$c = h_1(k) = Ak^{\alpha} \quad \text{and} \quad \widetilde{k} = g(k, h_1(k)) = 0$$
- Inserting the policy function $h_1(k)$ into the RHS of (8) leads to
$$V_1(k) = \ln(\underbrace{Ak^{\alpha}}_{h_1(k)}) + \beta \underbrace{V_0[g(k, h_1(k))]}_{=0} = \underbrace{\ln(A)}_{a_1} + \alpha \ln(k)$$

III Example: Value function vs. Lagrange approach

Solution via value function iteration

Value function iteration: **Second step** ($j = 1$)

- Use the just derived function $V_1(k)$,ie

$$V_1(k) = a_1 + \alpha \ln(k)$$

- Max of RHS of (8) over c requires

$$\frac{\partial[\ln(c) + \beta V_1[Ak^\alpha - c]]}{\partial c} = \frac{\partial[\ln(c) + \beta[a_1 + \alpha \ln(Ak^\alpha - c)]}{\partial c} = 0,$$

  implying

$$\frac{1}{c} = \alpha\beta \frac{1}{Ak^\alpha - c},$$

  ie we get

$$c = h_2(k) = \frac{1}{1+\alpha\beta} Ak^\alpha \quad \text{and} \quad \widetilde{k} = g(k, h_2(k)) = \frac{\alpha\beta}{1+\alpha\beta} Ak^\alpha$$

- Inserting the policy function $h_2(k)$ into the RHS of (8) leads to

$$V_2(k) = \ln(\underbrace{\frac{1}{1+\alpha\beta} Ak^\alpha}_{h_2(k)}) + \beta[a_1 + \alpha \ln(\underbrace{\frac{\alpha\beta}{1+\alpha\beta} Ak^\alpha}_{g(k, h_2(k))})] = a_2 + (1+\alpha\beta)\alpha \ln(k)$$

III Example: Value function vs. Lagrange approach

Solution via value function iteration

Value function iteration: **Third step** $(j = 2)$

- Use the just derived function $V_2(k) = a_2 + (1+\alpha\beta)\alpha\ln(k)$
- Max of RHS of (8) over c requires

  $$\frac{\partial[\ln(c) + \beta V_2[Ak^\alpha - c]]}{\partial c} = \frac{\partial[\ln(c) + \beta[a_2 + (1+\alpha\beta)\alpha\ln(Ak^\alpha - c)]}{\partial c} = 0,$$

  implying

  $$\frac{1}{c} = \alpha\beta[1+\alpha\beta]\frac{1}{Ak^\alpha - c},$$

  ie we get

  $$c = h_3(k) = \frac{1}{1+\alpha\beta+(\alpha\beta)^2}Ak^\alpha \quad \text{and} \quad \widetilde{k} = g(k, h_3(k)) = \frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}Ak^\alpha$$

- Inserting the policy function $h_3(k)$ into the RHS of (8) leads to

$$\begin{aligned} V_3(k) &= \ln(\underbrace{\frac{Ak^\alpha}{1+\alpha\beta+(\alpha\beta)^2}}_{h_3(k)}) + \beta[a_2 + (1+\alpha\beta)\alpha\ln(\underbrace{\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}Ak^\alpha}_{g(k,h_3(k))})] \\ &= a_3 + [1+\alpha\beta+(\alpha\beta)^2]\alpha\ln(k) \end{aligned}$$

III Example: Value function vs. Lagrange approach

Solution via value function iteration

Value function iteration: **Convergence ($j \to \infty$)**

- **Policy function:** consider what we got from the iterations done so far:

$$\begin{aligned} h_1(k) &= Ak^\alpha \\ h_2(k) &= \frac{1}{1+\alpha\beta}Ak^\alpha \\ h_3(k) &= \frac{1}{1+\alpha\beta+(\alpha\beta)^2}Ak^\alpha \end{aligned}$$

- There is a pattern behind this, ie after $j = T$ steps we will get

$$h_T(k) = \frac{1}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3+\ldots+(\alpha\beta)^{T-1}}Ak^\alpha$$

- Recall: $\alpha \in (0,1)$, $\beta \in (0,1)$, implying $\alpha \cdot \beta \in (0,1)$
- Thus, the iteration process ensures that the policy function converges:

$$c = h_\infty(k) = h(k) = (1-\alpha\beta)Ak^\alpha \tag{10}$$

- Similarly

$$\widetilde{k} = g(k, h_\infty(k)) = g(k, h(k)) = \alpha\beta Ak^\alpha \tag{11}$$

III Example: Value function vs. Lagrange approach

Solution via value function iteration

Value function iteration: **Convergence ($j \rightarrow \infty$)**

- **Value function:** consider what we got from the iterations done so far:

$$\begin{aligned} V_0(k) &= 0 \\ V_1(k) &= a_1 + \alpha \ln(k) \\ V_2(k) &= a_2 + (1 + \alpha\beta)\alpha \ln(k) \\ V_3(k) &= a_3 + [1 + \alpha\beta + (\alpha\beta)^2]\alpha \ln(k) \end{aligned}$$

- As concerns the terms including $\alpha \ln(k)$, there is a pattern behind this, ie after $j = T$ steps we will get

$$V_T(k) = a_T + [1 + \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3 + ... + (\alpha\beta)^{T-1}]\alpha \ln(k)$$

- Since $\alpha \cdot \beta \in (0, 1)$, for $j \rightarrow \infty$ there will be convergence, ie

$$V_\infty(k) = V(k) = a_\infty + \frac{\alpha}{1 - \alpha\beta} \ln(k)$$

- What is still missing before we can fully characterize $V(k)$?
  $\rightarrow$ The limit term

$$a = a_\infty$$

III Example: Value function vs. Lagrange approach

Solution via value function iteration

Value function iteration: **Convergence ($j \to \infty$)**

- To find the limit term

$$a = a_\infty$$

  we could carefully exploit the **algebra of geometric series** to find a converging pattern behind $a_1, a_2, a_3, ...a_T ...$
- Alternatively, we can use the **method** of **undetermined coefficients**
- Consider eqn (9), ie

$$V(k) = \ln(h(k)) + \beta V[g(k, h(k))],$$

  which will be satisfied as $j \to \infty$. Combining the so far established limit values we can write this eqn as

$$V(k) = a + \frac{\alpha}{1-\alpha\beta}\ln(k) = \ln[\underbrace{(1-\alpha\beta)Ak^{\alpha}]}_{h(k)} + \beta[a + \frac{\alpha}{1-\alpha\beta}\ln\underbrace{(\alpha\beta Ak^{\alpha})}_{g(k,h(k))}]$$

- Within this eqn we can determine a by combining all those terms which are not linked to $\alpha\ln(k)$...

III Example: Value function vs. Lagrange approach

Solution via value function iteration

Value function iteration: **Convergence ($j \to \infty$)**

- ...ie from

$$V(k) = a + \frac{\alpha}{1-\alpha\beta}\ln(k) = \ln[\underbrace{(1-\alpha\beta)Ak^{\alpha}}_{h(k)}] + \beta[a + \frac{\alpha}{1-\alpha\beta}\ln(\underbrace{\alpha\beta Ak^{\alpha}}_{g(k,h(k)})]$$

  we can obtain (via elimination of terms linked to $\alpha\ln(k)$)

$$a = \ln[(1-\alpha\beta)A] + a\beta + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta A)$$

  such that the value of a can be determined as

$$a = \frac{1}{1-\beta}\left(\ln[(1-\alpha\beta)A] + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta A)\right)$$

- In sum, using this expression for a, the fully determined **value function** $V(k)$ is given by

$$V(k) = \underbrace{\frac{1}{1-\beta}\left(\ln[(1-\alpha\beta)A] + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta A)\right)}_{a} + \frac{\alpha}{1-\alpha\beta}\ln(k) \quad (12)$$

III Example: Value function vs. Lagrange approach

Comparison with Lagrange solution

- The value function approach is one solution technique among many others
- Alternatively, the problem at hand can be solved with the Lagrange approach
- The value function approach is often used to implicitly characterize optimal solutions of problems for which no explicit solution exists
- Moreover, it is a convenient tool to obtain numerical solutions

$\longrightarrow$ Since this particular example does have a closed-form solution it is instructive to verify the relationship between the two approaches

III Example: Value function vs. Lagrange approach

Comparison with Lagrange solution

- Consider

$$\mathcal{L}_t = \sum_{t=0}^{\infty} \beta^t \{\ln(c_t) + \lambda_t [A k_t^{\alpha} - k_{t+1} - c_t]\}$$

- Optimization of $\mathcal{L}_t$ over the choice variables $\{c_t, k_{t+1}, \lambda_t; \forall t \geqslant 0\}$ leads to a two-dimensional, non-linear system of first-order difference equations in c and k, ie
  the **consumption Euler equation**

$$\frac{1}{c_t} = \underbrace{\beta \alpha A k_{t+1}^{\alpha-1}}_{f'(k_{t+1})} \frac{1}{c_{t+1}} \quad (13)$$

  and the **dynamic resource constraint**

$$k_{t+1} = A k_t^{\alpha} - c_t \quad (14)$$

- Moreover, the optimal sequences of variables are subject to the **initial condition** k_0 and the **terminal condition**

$$\lim_{T\to\infty} \beta^T \lambda_T k_{T+1} = \beta^T \frac{1}{c_T} k_{T+1} = 0$$

III Example: Value function vs. Lagrange approach

Comparison with Lagrange solution

→ To convince ourselves that the two approaches lead to equivalent outcomes we will undertake 3 comparisons:

- Comparison 1: **Transitional dynamics**
- Comparison 2: **Steady-state solution**
- Comparison 3: **Welfare derived from steady-state consumption**

III Example: Value function vs. Lagrange approach

Comparison with Lagrange solution

Comparison 1: Transitional dynamics

- Recall from the general set-up discussed above that the value function solution is characterized by eqns of type (6) and (7)
- Eqn (6), ie
$$u'(c) + \beta \frac{\partial g(k,c)}{\partial c} V'[g(k,c)] = 0,$$
results from maximizing the RHS of the Bellman equation w.r.t. to the control variable c. For our example, using the constraint
$$\widetilde{k} = g(k,c) = Ak^{\alpha} - c,$$
it is given by
$$\frac{1}{c} = \beta V'(\widetilde{k}) \tag{15}$$
- Eqn (7), ie
$$V'(k) = \beta \frac{\partial g(k,h(k))}{\partial k} V'[g(k,h(k))]$$
implicitly characterizes the optimality of the solution $V(k)$ via an envelope condition. For our example it is given by
$$V'(k) = \beta \alpha A k^{\alpha-1} V'(\widetilde{k}) \tag{16}$$

III Example: Value function vs. Lagrange approach

Comparison with Lagrange solution

- Consider the two eqns (15) and (16), ie

$$\frac{1}{c} = \beta V'(\widetilde{k})$$

$$V'(k) = \beta\alpha A k^{\alpha-1} V'(\widetilde{k})$$

- When forwarded by one period and using

$$\frac{1}{\widetilde{c}} = \beta V'(\widetilde{\widetilde{k}})$$

they can be combined to give the **consumption Euler equation** (13), ie

$$\frac{1}{c} = \beta \cdot \underbrace{\alpha A(\widetilde{k})^{\alpha-1}}_{f'(\widetilde{k})} \cdot \frac{1}{\widetilde{c}},$$

- In sum, the transitional dynamics of the Value function and the Lagrange solutions are characterized by the same difference equations

III Example: Value function vs. Lagrange approach
Comparison with Lagrange solution

Comparison 2: Long-run (steady-state) solution

- The consumption Euler equation and the dynamic resource constraint derived under the Lagrange approach, ie

$$\frac{1}{c_t} = \underbrace{\beta\alpha A k_{t+1}^{\alpha-1}}_{f'(k_{t+1})} \frac{1}{c_{t+1}} \quad \text{and} \quad k_{t+1} = A k_t^{\alpha} - c_t$$

  are characterized by a unique and saddlepath-stable steady state, with

$$k^* = (\alpha\beta A)^{\frac{1}{1-\alpha}}$$

$$c^* = A \cdot (k^*)^{\alpha} - k^* = A \cdot (k^*)^{\alpha} \cdot [1 - \frac{(k^*)^{1-\alpha}}{A}] = (1-\alpha\beta) \cdot A \cdot (k^*)^{\alpha}$$

- These steady state values are consistent with eqns (10) and (11) obtained from the value function iteration, ie

$$c = h(k) = (1-\alpha\beta) A k^{\alpha}$$

$$\widetilde{k} = g(k, h(k)) = \alpha\beta A k^{\alpha},$$

  where by concavity of k^{α} the values $\widetilde{k}$ and c converge against k^* and c^*

III Example: Value function vs. Lagrange approach

Comparison with Lagrange solution

Comparison 3: Welfare derived from steady-state consumption

- Assume the economy is in steady state, ie $k_0 = k^*$
- Then, the welfare of the representative consumer will be given by

$$V(k^*) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t) = \ln(c^*) \cdot \sum_{t=0}^{\infty} \beta^t,$$

amounting to

$$\begin{aligned} V(k^*) &= \frac{1}{1-\beta} \ln(c^*) \\ &= \frac{1}{1-\beta} \{\ln[(1-\alpha\beta)A] + \alpha \ln(k^*)\} \\ &= \frac{1}{1-\beta} \{\ln[(1-\alpha\beta)A] + \frac{\alpha}{1-\alpha} \ln(\alpha\beta A)\} \qquad (17) \end{aligned}$$

III Example: Value function vs. Lagrange approach

Comparison with Lagrange solution

Comparison 3: Welfare derived from steady-state consumption

- Recall the value function derived above in the general expression (12), ie

$$V(k) = \frac{1}{1-\beta}\{\ln[(1-\alpha\beta)A] + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta A)\} + \frac{\alpha}{1-\alpha\beta}\ln(k)$$

- Let $k = k^* = (\alpha\beta A)^{\frac{1}{1-\alpha}}$, implying

$$V(k^*) = \frac{1}{1-\beta}\ln[(1-\alpha\beta)A] + \{\frac{1}{1-\beta}\cdot\frac{\alpha\beta}{1-\alpha\beta} + \frac{1}{1-\alpha}\cdot\frac{\alpha}{1-\alpha\beta}\}\ln(\alpha\beta A) \tag{18}$$

- Comparing coefficients between $\ln(\alpha\beta A)$-related terms, eqns (17) and (18) will be identical if

$$\frac{1}{1-\beta}\cdot\frac{\alpha}{1-\alpha} = \frac{1}{1-\beta}\cdot\frac{\alpha\beta}{1-\alpha\beta} + \frac{1}{1-\alpha}\cdot\frac{\alpha}{1-\alpha\beta},$$

which is, indeed, the case

→ **Welfare** is **identical** under value function and Lagrange solutions

IV The basic MIU model: value function solution

- Recall from Lecture 2 that we used the Lagrange approach to derive the intertemporal optimality conditions which characterize the basic MIU model
- By contrast, Walsh (Chapter 2) derives these conditions using the value function approach
- In view of the techniques introduced in this lecture it should be no surprise why the two approaches generate identical results

IV The basic MIU model: value function solution

→ Recall from Lecture 2 the main ingredients of the intertemporal optimization problem of the representative household

- Utility function to be maximized via optimal choices of c_t, m_t, b_t, k_t:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, m_t)$$

- Budget constraint (in per capita terms):

$$f\left(\frac{k_{t-1}}{1+n}\right) + \tau_t + (1-\delta)\frac{k_{t-1}}{1+n} + \frac{(1+i_{t-1})b_{t-1} + m_{t-1}}{(1+n)(1+\pi_t)} = c_t + k_t + b_t + m_t$$

- Maximization is subject to initial and terminal conditions: k_{-1} is predetermined and $\lim_{t\to\infty} \beta^t u_{c,t} x_t = 0 \quad x = k, b, m$

IV The basic MIU model: value function solution

- To solve this problem via the value function approach it is convenient to introduce a new **state variable** ω_t which summarizes all **resources** of the representative HH at the beginning of period t:

$$\omega_t \equiv f(\frac{k_{t-1}}{1+n}) + \tau_t + (1-\delta)\frac{k_{t-1}}{1+n} + \frac{(1+i_{t-1})b_{t-1} + m_{t-1}}{(1+n)(1+\pi_t)} = c_t + k_t + b_t + m_t$$

- Using this definition of ω, we can substitute out for the (old) state variable k, ie

$$k_t = \omega_t - c_t - b_t - m_t$$

- If one combines the last two eqns, the **law of motion** of the **state variable** ω can be arranged to satisfy the structure

$$\omega_{t+1} = g(\omega_t, c_t, b_t, m_t), \quad \text{ie}$$

$$\begin{aligned} \omega_{t+1} &= f(\frac{\omega_t - c_t - b_t - m_t}{1+n}) + \tau_{t+1} \\ &\quad + (1-\delta)\frac{\omega_t - c_t - b_t - m_t}{1+n} + \frac{(1+i_t)b_t + m_t}{(1+n)(1+\pi_{t+1})} \qquad (19) \end{aligned}$$

IV The basic MIU model: value function solution

- Using (19), the value function satisfies

$$V(\omega) = \max_{c,b,m} \{u(c,m) + \beta V(\widetilde{\omega})\} \tag{20}$$

subject to

$$\begin{aligned} \widetilde{\omega} &= g(\omega, c, m, b) \\ &= f\left(\frac{\omega - c - b - m}{1+n}\right) + \widetilde{\tau} + \frac{1-\delta}{1+n}(\omega - c - b - m) + \frac{(1+i)b + m}{(1+n)(1+\widetilde{\pi})} \end{aligned}$$

and with ω given

- *Notice:* $\widetilde{\tau}$, $\widetilde{\pi}$, and i are exogenously given for the representative HH

IV The basic MIU model: value function solution

- For suitable assumptions on functional forms (see the corresponding discussion in Lecture 2), eqn (20) is solved by a unique value function $V(\omega)$, with associated unique policy functions for the control variables c, b, and m

- To characterize the behaviour of the optimal solution, let us derive the optimality conditions
  i) for the for control variables c, b, and m in line with eqn (6) and
  ii) for the state variable ω in line with eqn (7)

IV The basic MIU model: value function solution

Consider the value function (20), ie

$$V(\omega) = \max_{c,b,m} \{u(c,m) + \beta V(\widetilde{\omega})\} \quad \text{s.t.}$$

$$\widetilde{\omega} = f\left(\frac{\omega - c - b - m}{1+n}\right) + \widetilde{\tau} + \frac{1-\delta}{1+n}(\omega - c - b - m) + \frac{(1+i)b + m}{(1+n)(1+\widetilde{\pi})}$$

- **Optimal choice of c:**

$$u_c(c,m) + \beta \frac{\partial \widetilde{\omega}}{\partial c} V'(\widetilde{\omega}) = 0$$

$$u_c(c,m) = \beta \left[\frac{1}{1+n} \cdot [f'(k') + 1 - \delta] \right] \cdot V'(\widetilde{\omega}) \tag{21}$$

- **Optimal choice of b:**

$$\beta \frac{\partial \widetilde{\omega}}{\partial b} V'(\widetilde{\omega}) = 0$$

$$\beta \left[-\frac{1}{1+n}[f'(k') + 1 - \delta] + \frac{1+i}{(1+n)(1+\widetilde{\pi})} \right] \cdot V'(\widetilde{\omega}) = 0 \tag{22}$$

IV The basic MIU model: value function solution

Consider the value function (20), ie

$$V(\omega) = \max_{c,b,m} \{u(c,m) + \beta V(\widetilde{\omega})\} \quad \text{s.t.}$$

$$\widetilde{\omega} = f(\frac{\omega - c - b - m}{1+n}) + \widetilde{\tau} + \frac{1-\delta}{1+n}(\omega - c - b - m) + \frac{(1+i)b + m}{(1+n)(1+\widetilde{\pi})}$$

- **Optimal choice of m:**

$$u_m(c,m) + \beta \frac{\partial \widetilde{\omega}}{\partial m} V'(\widetilde{\omega}) = 0$$

$$u_m(c,m) = \beta \cdot \left[\frac{1}{1+n}[f'(k') + 1 - \delta] - \frac{1}{(1+n)(1+\widetilde{\pi})} \right] \cdot V'(\widetilde{\omega}) \quad (23)$$

- **Optimal choice of ω:**

$$V'(\omega) = \beta \frac{\partial \widetilde{\omega}}{\partial \omega} V'(\widetilde{\omega})$$

$$V'(\omega) = \beta \left[\frac{1}{1+n}[f'(k') + 1 - \delta] \right] \cdot V'(\widetilde{\omega}) \quad (24)$$

IV The basic MIU model: value function solution

→ Let us use the definition of the real interest rate

$$1 + r = 1 + f'(k') - \delta$$

and combine the four optimality conditions (21)-(24) to eliminate the terms $V'(\omega)$ and $V'(\widetilde{\omega})$:

- Combining (21) and (24), ie

$$u_c(c, m) = \beta \cdot (\frac{1+r}{1+n}) \cdot V'(\widetilde{\omega}) \quad \text{and} \quad V'(\omega) = \beta \cdot (\frac{1+r}{1+n}) \cdot V'(\widetilde{\omega})$$

gives

$$u_c(c, m) = V'(\omega)$$

and, accordingly,

$$u_c(c, m) = \beta \cdot (\frac{1+r}{1+n}) \cdot u_c(\widetilde{c}, \widetilde{m}) \tag{25}$$

- Eqn (22) implies

$$1 + r = \frac{1+i}{1+\widetilde{\pi}} \tag{26}$$

IV The basic MIU model: value function solution

- Combining (21) and (23), ie

$$
\begin{aligned}
u_c(c,m) &= \beta\left[\frac{1}{1+n}\cdot[f'(k')+1-\delta]\right]\cdot V'(\widetilde{\omega}) \\
u_m(c,m) &= \beta\cdot\left[\frac{1}{1+n}[f'(k')+1-\delta]-\frac{1}{(1+n)(1+\widetilde{\pi})}\right]\cdot V'(\widetilde{\omega})
\end{aligned}
$$

leads to:

$$
\begin{aligned}
u_m(c,m) &= \beta\cdot\frac{1+r}{1+n}\cdot V'(\widetilde{\omega})-\beta\frac{1}{(1+n)(1+\widetilde{\pi})}\cdot V'(\widetilde{\omega}) \\
&= [1-\frac{1}{(1+r)(1+\widetilde{\pi})}]\cdot u_c(c,m) \\
&= \frac{i}{1+i}u_c(c,m) \qquad (27)
\end{aligned}
$$

IV The basic MIU model: value function solution

Summary: Consistent with the system of intertemporal optimality conditions derived in Lecture 2 under the Lagrange approach, eqns (25), (26), and (27) reproduce, respectively:

- **Consumption Euler equation**

$$u_c(c_t, m_t) = \beta \cdot \left(\frac{1 + r_t}{1 + n}\right) \cdot u_c(c_{t+1}, m_{t+1})$$

- **Fisher equation**

$$\underbrace{\left(1 + f'\left(\frac{k_{t-1}}{1 + n}\right) - \delta\right)}_{1+r_t} \cdot (1 + \pi_{t+1}) = 1 + i_t$$

- **Optimal allocation rule for real balances**

$$u_m(c_t, m_t) = \frac{i_t}{1 + i_t} u_c(c_t, m_t)$$

Moreover, the **resource constraint** closes the system by accounting for the dynamics of the capital stock, ie

$$c_t + k_t = f\left(\frac{k_{t-1}}{1 + n}\right) + (1 - \delta)\frac{k_{t-1}}{1 + n}$$