1 Life is live

Individuals live their life from birth at time $\tau = 0$ until death at time T and experience utility given by

$$U(T) \equiv \int_0^T \theta(\tau) u(c(\tau)) d\tau,$$

where $u(c(\tau))$ is explicitly given by

$$u(c(\tau)) = \frac{c(\tau)^{1-\gamma} - 1}{1-\gamma}$$

and the discount factor $\theta(\tau)$ satisfies

$$\rho(\tau) \equiv -\dot{\theta}(\tau)/\theta(\tau). \tag{1}$$

But individuals are actually uncertain about the exact length T of their life. Let $f(T)$ denote the probability density function of the random variable T with support $[0, \Omega]$, where Ω is the finite maximum possible lifetime.

1. Deriving the expected utility function

   (a) Define expected utility of an individual that is uncertain about the length of her life.

   (b) Derive an explicit expression for θ by solving the differential equation 1 with initial condition $\theta(0) = 1$.

   (c) Show that expected utility can be expressed by equation 2. To this end, consider $\int_0^\Omega f(T)U(T)dT$ and apply integration by parts using (equation (4.3.5) in the lecture notes)

   $$\int_a^b \dot{x} g dt = [xg]_a^b - \int_a^b x \dot{g} dt.$$

   You can use the definition

   $$p(T) \equiv \int_T^\Omega f(z) dz$$

   which denotes the probability at birth that an individual will be alive at age T.

2. Use the current value Hamiltonian to maximize the objective function

$$\int_0^\Omega e^{-\int_0^\tau \rho(x)dx} p(\tau) u(c(\tau)) d\tau, \tag{2}$$

subject to

$$\dot{a}(\tau) = \left[r - \delta \frac{\dot{p}(\tau)}{p(\tau)}\right] a(\tau) + w(\tau) - c(\tau), \tag{3}$$

$$a(0) = 0, \quad a(\Omega) \geq 0. \tag{4}$$

3. Compute optimal wealth $a(\tau)$ by solving the differential equation (3) using the boundary conditions (4).

4. Derive an explicit expression for $c(\tau)$ by solving the differential equation

$$\dot{c}(\tau) = \sigma \left(r + (1-\delta) \frac{\dot{p}(\tau)}{p(\tau)} + \frac{\dot{\theta}(\tau)}{\theta(\tau)} \right) c(\tau).$$

Use optimal wealth $a(\tau)$ to derive an expression for $c(0)$.

2 Life is full of choices

The individual's utility function is given by a Kreps-Porteus structure,

$$U(c_0, \tilde{c}_1) = u_0(c_0) + u_1(v^{-1}(E[v(\tilde{c}_1)]), \tag{5}$$

where u_0, u_1 and v are three increasing functions given by

$$v(z) = \frac{z^{1-\gamma}}{1-\gamma} \quad \text{and} \quad u_1(z) = \beta u_0(z) = \frac{z^{1-\alpha}}{1-\alpha}. \tag{6}$$

1. Derive an explicit expression for the utility function 5 using the functions in 6
2. Maximize the utility function

$$U(c_0, \tilde{c}_1) = \phi \left[c_0^{1-\alpha} + \beta \left(E[\tilde{c}_1^{1-\gamma}] \right)^{\frac{1-\alpha}{1-\gamma}} \right], \tag{7}$$

   (where ϕ is a constant) for $\alpha = \gamma$ subject to the budget constraints

$$\begin{aligned} c_0 &= w_0 - s, \\ \tilde{c}_1 &= w_1 + \tilde{x} + \rho s. \end{aligned}$$

3. Now, maximize the utility function 7 subject to the budget constraints

$$\begin{aligned} c_0 &= w_0 - s, \\ \tilde{c}_1 &= w_1 + \tilde{x} + \rho s, \end{aligned}$$

   but this time without assuming $\alpha = \gamma$.

4. Compare the results for $\alpha = \gamma$ and $\alpha \neq \gamma$.