Part I
Modelling Money in General Equilibrium: a Primer
Lecture 2
The Basic MIU Model

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I Model ingredients

Features of the basic MIU Model (*Walsh, Section 2.2*)

- flexible prices
- deterministic set-up
- perfect foresight
- no labour supply decision, ie per capita labour supply is fixed at $n^{ls} \equiv 1$
- exogenous and constant population growth: $N_t = (1+n)N_{t-1},\ n \geqslant 0$

I Model ingredients

Objective of representative household:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, m_t) \quad \beta \in (0,1) \tag{1}$$

Properties of flow utility $u(c_t, m_t)$:

- continuously differentiable, increasing in both arguments, and strictly concave
- **(A 1):** *sufficient* (and mild) condition to ensure a monetary equilibrium with $m_t > 0$:
  (i) $u_m(c,m)|_{m=0} \to \infty \ \forall c > 0$,
  (ii) there exists some (possibly large) satiation value of m such that $u_m(c,m)|_{m=\overline{m}} = 0 \ \forall c > 0$
  ($\rightarrow$ below we consider variations of (A1))

I Model ingredients

Technology:
Neoclassical aggregate production function with

$$Y_t = F(K_{t-1}, N_t)$$

- In period t, aggregate output Y_t is a function F of two inputs: contemporaneous labour (N_t) and predetermined capital (K_{t-1})
- Function F has constant returns to scale
- Per capita output $(y_t \equiv \frac{Y_t}{N_t})$:

$$y_t = \frac{F(K_{t-1}, N_t)}{N_t} = F(\frac{K_{t-1}}{N_t}, 1) \equiv f(\frac{k_{t-1}}{1+n}) = f(k'_{t-1}) \text{ with: } k'_{t-1} \equiv \frac{k_{t-1}}{1+n}$$

(A 2): Properties of per capita output $y = f(k')$:

- f is continuously differentiable, $f_k(k') > 0$, $f_{kk}(k') < 0$
- Inada conditions: (i) $f_k(k'))|_{k'=0} \to \infty$, (ii) $f_k(k'))|_{k' \to \infty} = 0$

I Model ingredients

Aggregate private sector budget constraint in real terms:

$$Y_t + \tau_t N_t + (1-\delta)K_{t-1} + \frac{(1+i_{t-1})B_{t-1} + M_{t-1}}{P_t} = C_t + K_t + \frac{B_t + M_t}{P_t}$$

τ_t : Per capita lump-sum transfer

B_{t-1} : Nominal amount of aggregate government bonds;
bought in period $t-1$; paying out $(1+i_{t-1})B_{t-1}$ in period t,
$i_{t-1} \geqslant 0$: nominal interest rate on gov't bonds, assumed to be non-negative

M_{t-1} : Nominal amount of aggregate money holdings;
'bought' in period $t-1$; paying out M_{t-1} in period t,
$i^M_{t-1} \equiv 0$: *nominal interest rate on (outside) money is zero*

P_t : aggregate price level in period t of the single economy-wide good

I Model ingredients

Per capita private sector budget constraint in real terms:

Dividing the previous equation by N_t yields:

$$f(\frac{k_{t-1}}{1+n}) + \tau_t + (1-\delta)\frac{k_{t-1}}{1+n} + \frac{(1+i_{t-1})b_{t-1} + m_{t-1}}{(1+n)(1+\pi_t)} = c_t + k_t + b_t + m_t \quad (2)$$

with:

- $b_t = \frac{B_t}{P_t N_t}$, $m_t = \frac{M_t}{P_t N_t}$
- inflation defined as $\frac{P_t}{P_{t-1}} \equiv 1 + \pi_t$
- and using:

$$\frac{(1+i_{t-1})B_{t-1}}{P_t N_t} = \frac{(1+i_{t-1})}{(1+n)N_{t-1}}\frac{B_{t-1}}{P_{t-1}}\frac{P_{t-1}}{P_t} = \frac{(1+i_{t-1})b_{t-1}}{(1+n)(1+\pi_t)}$$

$$\frac{M_{t-1}}{P_t N_t} = \frac{1}{(1+n)N_{t-1}}\frac{M_{t-1}}{P_{t-1}}\frac{P_{t-1}}{P_t} = \frac{m_{t-1}}{(1+n)(1+\pi_t)}$$

$\rightarrow$ From now on, define the real interest rate as:

$$1 + r_{t-1} = \frac{1+i_{t-1}}{1+\pi_t}$$

I Model ingredients

Per capita government budget constraint in real terms:

$$\tau_t + \frac{1 + r_{t-1}}{1+n} b_{t-1} + \frac{1}{(1+n)(1+\pi_t)} m_{t-1} = b_t + m_t \qquad (3)$$

Write equivalently as:

$$\tau_t + \frac{1 + r_{t-1}}{1+n} b_{t-1} = b_t + \underbrace{m_t - \frac{1}{(1+n)(1+\pi_t)} m_{t-1}}_{\text{Seigniorage}}$$

Simplifying assumptions:

- no government consumption ($g_t \equiv 0$) or government investment
- no distortionary (regular) taxes
  ($\longrightarrow$ *to be removed in Part II of the Lecture*)
- τ_t adjusts endogenously to balance (3) $\forall t \geqslant 0$

II Solution based on Lagrange multipliers

- **Characterization of competitive equilibrium** requires, inter alia, to solve an intertemporal optimization of the representative household
- To solve such problems (here: in discrete time) various techniques exist
- We solve the problem by the **Lagrange multiplier approach**
- Later we will verify that the **value function approach** used by Walsh leads to the same results
- in case you find continuous time ‘easier’:
  $\longrightarrow$ good treatment of MIU-model in Blanchard and Fisher (1989)!

$\longrightarrow$ Next slide: overview of maximization problem of representative household and the first-order conditions (FOCs) of an interior optimum

II Solution based on Lagrange multipliers

Maximize (1) s.t. budget constraint (2) over c_t, m_t, b_t, k_t:

$$\max \sum_{t=0}^{\infty} \beta^t \Big[u(c_t, m_t)$$

$$+\lambda_t \{ f(\frac{k_{t-1}}{1+n}) + \tau_t + (1-\delta)\frac{k_{t-1}}{1+n} + \frac{(1+i_{t-1})b_{t-1} + m_{t-1}}{(1+n)(1+\pi_t)} - c_t - k_t - b_t - m_t \} \Big]$$

FOCs (interior) w.r.t. c_t, m_t, b_t, k_t ($\forall t \geqslant 0$):

$$u_c(c_t, m_t) - \lambda_t = 0 \qquad (4)$$

$$u_m(c_t, m_t) - \lambda_t + \beta\lambda_{t+1}\frac{1}{(1+n)(1+\pi_{t+1})} = 0 \qquad (5)$$

$$-\lambda_t + \beta\lambda_{t+1}\frac{1+i_t}{(1+n)(1+\pi_{t+1})} = 0 \qquad (6)$$

$$-\lambda_t + \beta\lambda_{t+1}\frac{f_k(k_t') + 1 - \delta}{1+n} = 0 \qquad (7)$$

Transversality condition: $\lim_{t\to\infty} \beta^t \lambda_t x_t = 0 \quad x = k, b, m \qquad (8)$

λ_t : shadow value of period t income (in terms of utility of period t)
$\beta^t \lambda_t$: shadow value of period t income (in terms of utility of period 0)

II Solution based on Lagrange multipliers

Elimination of λ_t and λ_{t+1} in the FOCs yields:

- From (6), (7): **Arbitrage condition between physical capital and real bonds** (assumed to be perfect substitutes)

$$1 + r_t = 1 + f_k(k_t') - \delta \tag{9}$$

leading to the **Fisher equation**

$$1 + i_t = (1 + f_k(k_t') - \delta)(1 + \pi_{t+1}) \tag{10}$$

- From (4), (6): Intertemporal consumption optimality (**Euler equation**)

$$u_c(c_t, m_t) = \beta \frac{1 + r_t}{1 + n} u_c(c_{t+1}, m_{t+1}) \tag{11}$$

- From (4)-(6): Intratemporal **optimal allocation between consumption and real balances**

$$\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = \frac{i_t}{1 + i_t} \tag{12}$$

where $\frac{i_t}{1+i_t}$ measures the opportunity cost of holding money

II Solution based on Lagrange multipliers

Interpretation of (12): **'Opportunity cost of holding money'**

$\longrightarrow$ How to optimally allocate one extra euro between real balances and consumption in period t?

- in period t, 1 extra Euro makes up $\frac{1}{p_t}$ units of real balances, yielding $\frac{1}{p_t}u_m(c_t, m_t)$ marginal utility
- since money is dominated in return by bonds, there is an opportunity cost to this, ie one loses $\frac{i_t}{p_{t+1}}$ units of period-$t+1$ goods. When discounted this amounts to a loss of $\frac{i_t}{p_{t+1}(1+r_t)}$ period-t goods and an associated marginal loss of $\frac{i_t}{p_{t+1}(1+r_t)}u_c(c_t, m_t)$ utility

$\longrightarrow$ Equating $\frac{1}{p_t}u_m(c_t, m_t)$ and $\frac{i_t}{p_{t+1}(1+r_t)}u_c(c_t, m_t)$ yields eq (12), ie

$$u_m(c_t, m_t) = \frac{i_t}{1+i_t}u_c(c_t, m_t)$$

II Solution based on Lagrange multipliers

Competitive equilibrium:

- representative household takes all prices as given
- prices settle down at values such that all markets clear and resulting allocations are consistent with individually optimal behaviour

Implication: combination of budget constraints of the private sector and of the government yields the resource constraint of the economy, ie combine

$$f\left(\frac{k_{t-1}}{1+n}\right)+\tau_t+(1-\delta)\frac{k_{t-1}}{1+n}+\frac{(1+i_{t-1})b_{t-1}+m_{t-1}}{(1+n)(1+\pi_t)}=c_t+k_t+b_t+m_t$$

and

$$\tau_t+\frac{1+r_{t-1}}{1+n}b_{t-1}+\frac{1}{(1+n)(1+\pi_t)}m_{t-1}=b_t+m_t$$

to obtain the (per capita) **resource constraint**

$$f\left(\frac{k_{t-1}}{1+n}\right)+(1-\delta)\frac{k_{t-1}}{1+n}=c_t+k_t \tag{13}$$

II Solution based on Lagrange multipliers

Comments: How to read equations (4)-(8)?

- necessary conditions for optimality (and sufficient conditions come from A1 and A2)
- concept of optimality applies to sequences of variables, ie (4)-(8) form a system of difference equations characterizing the behaviour of the competitive equilibrium over time
- crucial for the exact time paths of variables consistent with such system: **initial** and **terminal** conditions

II Solution based on Lagrange multipliers

Remark 1: Initial conditions

- **Assumption (A 3):** The economy starts to operate in $t = 0$, taken as given the exogenous sequence $\{N_t\}$, the predetermined **real** value K_{-1} as well as the **nominal** values M_{-1}, B_{-1}, i_{-1}

$\rightarrow$ *This distinction between nominal and real initial values has implications for the (per capita) dynamics of the system of equilibrium equations:*

- Capital (k) is a state variable (with predetermined initial value k_{-1})
- Gov't liabilities (m, b) are not state variables, since the real value of $M_{-1} + (1 + i_{-1})B_{-1}$ in terms of period-0 goods, ie $\frac{M_{-1}+(1+i_{-1})B_{-1}}{P_0}$ is not predetermined.
  Why ? the period-0 price level P_0 is *not* predetermined, ie
  P_0 is determined within the competitive equilibrium, beginning in $t = 0$
- c is not a state variable, since c_{-1} does not enter any of the equations

$\rightarrow$ k is the single **predetermined (state) variable**
$\rightarrow$ other variables are **forwardlooking (control) variables** w/o initial conditions
$\rightarrow$ this feature becomes important below (when we discuss stability issues)

II Solution based on Lagrange multipliers

Remark 2: Terminal conditions

- The transversality condition (8) closes the system by backward induction from the (distant) future
- **Intuition:** consider for some future period $T > 0$ the terms $\beta^T \lambda_T x_T \quad (x = k, b, m)$. They describe the present value of the utility that could be obtained if the assets get consumed at T rather than invested
- If T is the terminal period it cannot be optimal, not to consume everything at T
- Infinite horizon analogy: As $T \longrightarrow \infty$, it cannot be optimal to postpone consumption forever, ie $\lim_{T \to \infty} \beta^T \lambda_T x_T = 0 \quad x = k, b, m$

III Core steady state features

From now on, consider 3 simplifying assumptions:

I) Constant population size

- $n = 0$, ie $N_t = N$, $\forall t \geqslant 0$

II) Zero level of equilibrium government bonds

- $B_t = 0$, $\forall t \geqslant 0$
  $\longrightarrow$ *Why is this assumption unproblematic?*

III) Constant money growth rule

- $M_t = (1 + \theta)M_{t-1}$, $\forall t \geqslant 0$, with $\theta \geq \underline{\theta}$
  (in the examples analyzed below we will assume $\theta \geq 0$)

III Core steady state features

Implications of III) of constant money growth, ie $M_t = (1+\theta)M_{t-1}$:

- Write the **law of motion of the inflation rate** as

$$1+\pi_{t+1} = \frac{P_{t+1}}{P_t} = \frac{M_t}{P_t}\frac{P_{t+1}}{M_{t+1}}(1+\theta) = \frac{m_t}{m_{t+1}}(1+\theta) \qquad (14)$$

  implying that in steady states, satisfying $m > 0$, we have

$$1+\pi = 1+\theta$$

- Similarly, write the **law of motion of the nominal interest rate** as

$$1+i_t = \underbrace{\left(1+f_k(k_t)-\delta\right)}_{1+r_t}\frac{m_t}{m_{t+1}}(1+\theta) \qquad (15)$$

  implying that in steady states, satisfying $m > 0$, we have

$$1+i = (1+r)(1+\theta)$$

III Core steady state features

Summary of intertemporal equilibrium conditions:
Using (14) and (15), rewrite (11), (12), and (13) as:

Euler equation:

$$\beta\underbrace{(1 + f_k(k_t) - \delta)}_{1+r_t} u_c(c_{t+1}, m_{t+1}) = u_c(c_t, m_t) \tag{16}$$

Resource constraint:

$$c_t + k_t = f(k_{t-1}) + (1 - \delta)k_{t-1} \tag{17}$$

Allocation between consumption and real balances:

$$\frac{1}{(1+\theta)(1+r_t)} u_c(c_t, m_t) \cdot m_{t+1} = [u_c(c_t, m_t) - u_m(c_t, m_t)] \cdot m_t \tag{18}$$

III Core steady state features

Summary of steady state conditions:
Consider the preceding 3 equations in steady state

Euler equation:

$$\beta \cdot \underbrace{(1+r)}_{1+f_k(k)-\delta} = 1 \quad \Leftrightarrow \quad f_k(k) = \frac{1}{\beta} - 1 + \delta \tag{19}$$

Resource constraint:

$$c = f(k) - \delta k \tag{20}$$

Allocation between consumption and real balances:

$$\frac{\beta}{1+\theta} u_c(c,m) \cdot m = [u_c(c,m) - u_m(c,m)] \cdot m \tag{21}$$

III Core steady state features

Existence of steady state:

$$\begin{aligned} f_k(k) &= \frac{1}{\beta} - 1 + \delta \\ c &= f(k) - \delta k \\ \frac{\beta}{1+\theta} u_c(c,m) \cdot m &= [u_c(c,m) - u_m(c,m)] \cdot m \end{aligned}$$

System has a recursive structure:

- 1st equation determines a unique value $k^* > 0$ (because of A 2)
- 2nd equation determines a unique value $c^*(k^*) > 0$
- 3rd equation: under mild assumptions (like A 1 and $\theta \geqslant \underline{\theta} \approx -r$), there exists $m^*(c^*, k^*) > 0$, satisfying $u_m = \frac{i}{1+i} u_c = (1 - \frac{\beta}{1+\theta}) u_c$ and respecting the 'zero lower bound constraint' $i \geqslant 0$

Steady-state government budget constraint ('behind the scenes'):

$$\tau = \frac{\theta}{1+\theta} m$$

III Core steady state features

Robust steady state features of the MIU model
→ It supports the dichotomy between real and nominal variables in terms of **neutrality** and **superneutrality**

$$\begin{aligned} f_k(k) &= \frac{1}{\beta} - 1 + \delta \\ c &= f(k) - \delta k \\ \frac{\beta}{1+\theta} u_c(c,m) \cdot m &= [u_c(c,m) - u_m(c,m)] \cdot m \end{aligned}$$

I) Neutrality (ΔM):

- The 3 equations are independent of the *level* of the nominal money stock M, ie they fix the variables k, y, c, r, m in real terms, and, for a given value of M, one obtains the price level $P = M/m$
- π and i are independent of the *level* of M
- a change in M leads to a proportionate change in the price level P

III Core steady state features

Robust steady state features of the MIU model

$$
\begin{aligned}
f_k(k) &= \frac{1}{\beta} - 1 + \delta \\
c &= f(k) - \delta k \\
\frac{\beta}{1+\theta} u_c(c,m) \cdot m &= [u_c(c,m) - u_m(c,m)] \cdot m
\end{aligned}
$$

II) Superneutrality ($\Delta\theta$):

- k, y, c, r are independent of the *growth rate* (θ) of the nominal money stock M
- a change in θ affects π and i, respectively, 'one-to-one' (using $i \approx r^* + \pi$)
- Moreover: since i captures the opportunity costs of holding money, a change in θ affects m via $u_m = \frac{i}{1+i} u_c$ (whenever $m > 0$)

III Core steady state features

Fragile features of the MIU model

I) Non-superneutrality during **transitional dynamics**

- Outside the steady state (during 'transitional dynamics'), superneutrality is, in general, not preserved
- Only under *very special* assumptions, like additively separable preferences in c and m, ie

$$u(c, m) = v(c) + \phi(m),$$

superneutrality prevails during the transitional dynamics
(to be discussed below)

III Core steady state features

Fragile features of the MIU model

II) Steady-state multiplicity

- if

$$u_m = \frac{i}{1+i} u_c = (1 - \frac{\beta}{1+\theta}) u_c$$

  has a unique positive solution $m^* > 0$, eq (21) may have a 2nd solution if we allow for the degenerate case of $m = 0$

- crucial in this context: structure of $u(c, m)$
- (famous) result by Obstfeld/Rogoff (1983):
  Assume $\theta \geqslant 0$ and consider $u(c, m) = v(c) + \phi(m)$.
  Then, the (seemingly) strong assumption:

$$(i)\ \phi_m(m)|_{m=0} \to \infty, (ii)\ \phi_m(m)|_{m\to\infty} = 0$$

  is *not* sufficient to rule out a 2nd steady state with $m_2^* = 0$

III Core steady state features

Fragile features of the MIU model

III) Stability

- (Saddle-path) Stability of 1st steady state with $m_1^* > 0$ cannot always be taken for granted in view of II):
  $\longrightarrow$ global stability issues under multiple steady states solutions!
- $\longrightarrow$ (remote?) possibility of a 'non-fundamental' (ie: solely speculative) **hyperinflation** in a world of pure fiat money, consistent, for example, with a constant money supply ($\theta = 0$) (see: Obstfeld/Rogoff, 1983)

IV Stability of steady states

Let us take these features as a motivation to do 2 things:
$\rightarrow$ i) understand the economic intuition behind them
$\rightarrow$ ii) be serious about backward and forward elements of solutions of systems of deterministic difference equations

Preview of what is to come below: 2 tractable example economies s.t.:

- 1) Non-negative money growth: $\theta \geqslant 0$
- 2) Cobb-Douglas production function: $y = k^\alpha$
- 3) Additively separable preferences: $u(c, m) = v(c) + \phi(m)$

(Standard) Example 1: $v(c) + \phi(m) = \log(c) + \log(m)$
$\rightarrow$ *to be shown:* unique steady state (with $m > 0$) and locally (saddle-path) stable dynamics
(Degenerate) Example 2: $v(c) + \phi(m) = \log(c) + \frac{1}{1-\sigma} m^{1-\sigma}$, $\sigma \in (0, 1)$
$\rightarrow$ *to be shown:* two steady states (with $m_1 > 0$, $m_2 = 0$), possibility of hyperinflationary dynamics converging against m_2

IV Stability of steady states

Special case: **recursive dynamics under additively separable preferences**
→ from now onwards, use $u(c, m) = v(c) + \phi(m)$ within (16)-(18):

Euler equation:

$$\beta \underbrace{(1 + f_k(k_t) - \delta)}_{1+r_t} v_c(c_{t+1}) = v_c(c_t) \tag{22}$$

Resource constraint:

$$c_t + k_t = f(k_{t-1}) + (1 - \delta) k_{t-1} \tag{23}$$

- -

Allocation between consumption and real balances:

$$B(c_t, k_t, m_{t+1}) \equiv \frac{1}{(1+\theta)(1+r_t)} v_c(c_t) \cdot m_{t+1} = [v_c(c_t) - \phi_m(m_t)] \cdot m_t \equiv A(c_t, m_t) \tag{24}$$

- (22) and (23) form a sub-system in c_t and k_t (ie independent of m_t)
- conditional on saddlepath-stability of (22)-(23), (in-)stability of the sequence m_t around (k^*, c^*) governed by the one-dimensional difference equation (24)

IV Stability of steady states

Recursive dynamics under additively separable preferences

$$\beta\underbrace{(1+f_k(k_t)-\delta)}_{1+r_t}\nu_c(c_{t+1}) = \nu_c(c_t)$$

$$c_t + k_t = f(k_{t-1}) + (1-\delta)k_{t-1}$$

$$B(c_t, k_t, m_{t+1}) \equiv \frac{1}{(1+\theta)(1+r_t)}\nu_c(c_t)\cdot m_{t+1} = [\nu_c(c_t) - \phi_m(m_t)]\cdot m_t \equiv A(c_t, m_t)$$

Transversality condition: $\lim_{t\to\infty}\beta^t\lambda_t x_t = 0 \quad x = k, b, m$

- 3 dynamic equations hold for all $t \geqslant 0$
  $\rightarrow$ 1st and 2nd equation have variables with index $t-1$, t, and $t+1$, but we can **transform** them to obtain a two-dimensional system of first-order difference equations

IV Stability of steady states

→ Use the **transformation**

$$c_t \equiv c_{t-1}^T$$

to replace the sub-system in c_t and k_t by the transformed sub-system in c_t^T and k_t s.t. $\forall t \geqslant -1$:

$$\beta\underbrace{(1 + f_k(k_{t+1}) - \delta)}_{1+r_{t+1}}\nu_c(c_{t+1}^T) = \nu_c(c_t^T) \tag{25}$$

$$c_t^T + k_{t+1} = f(k_t) + (1-\delta)k_t \tag{26}$$

notice: this transformation does not affect the sequence of events, ie the transformed system in c and k and the initial system are equivalent

→ Moreover, dynamics of (24) around a steady state with (k^*, c^*) satisfy

$$B(m_{t+1}) \equiv \frac{\beta}{1+\theta}\nu_c(c^*) \cdot m_{t+1} = [\nu_c(c^*) - \phi_m(m_t)] \cdot m_t \equiv A(m_t) \tag{27}$$

IV Stability of steady states

Notion of saddle-path stability

→ *Recall from above:*

- k is the single (backward-looking) state variable of the dynamic system (with predetermined initial value k_{-1})
- c and m are two (forward-looking) control variables w/o initial conditions

→ This feature is picked up by the notion of a **saddle-path stable solution** of the system (25)-(27)

→ **Idea:** combine the single initial condition k_{-1} and two terminal conditions (restricting c^T_{t+T} and m_{t+T}, assuming $T \to \infty$, and derived from the TV-condition) to find a solution of the form $(\forall t \geq -1)$)

$$k_{t+1} = \chi(k_t)$$

$$c^T_t = \xi_1(k_t), \quad m_t = \xi_2(k_t)$$

→ In general, the functions χ and ξ_1, ξ_2 will be non-linear. Approximate solutions rely on linear functions, characterizing a linearized version of the system (25)-(27)

IV Stability of steady states

Linearized dynamics

Recursive dynamics of the linearized system:

$\rightarrow$ The system (25)-(27) is non-linear. 'Way out'?

$\rightarrow$ Analysis of a linearized system, obtained from a 1st-order Taylor expansion of (25)-(27) around some steady state (k^*, c^*, m^*) :

$$\begin{bmatrix} c_{t+1}^T - c^* \\ k_{t+1} - k^* \end{bmatrix} = A \cdot \begin{bmatrix} c_t^T - c^* \\ k_t - k^* \end{bmatrix} \tag{28}$$

- -

$$m_{t+1} - m^* = a_m \cdot (m_t - m^*) \tag{29}$$

- A is a 2x2-matrix, with coefficients evaluated at the steady state, ie $A = \begin{bmatrix} a_{11}(k^*, c^*) & a_{12}(k^*, c^*) \\ a_{21}(k^*, c^*) & a_{22}(k^*, c^*) \end{bmatrix}$
- Similarly, a_m is a scalar, with $a_m = a_m(k^*, c^*, m^*)$

IV Stability of steady states
Graphical characterization

Dynamics of the linearized system: phase diagrams
$\longrightarrow$ The dynamics of this linearized system can be solved analytically
$\longrightarrow$ Without loss of generality we will consider a graphical representation of the stability behaviour, using phase diagrams

$\longrightarrow$ We do this for the 2 example economies, respectively, in two steps:

- **Step 1**: Calculation of steady states values (and check: unique vs. multiple steady states)
- **Step 2:** Construction of phase diagrams around the steady state with $m > 0$

IV Stability of steady states

Graphical characterization: example economy 1

Example 1: $\theta \geqslant 0$, $y = k^{\alpha}$, and $v(c) + \phi(m) = \log(c) + \log(m)$

Step I: steady state calculation

From (19), ie $f_k(k^*) = \frac{1}{\beta} - 1 + \delta = \alpha(k^*)^{\alpha-1}$:

$$k^* = \Big(\frac{\alpha\beta}{1-\beta+\delta\beta}\Big)^{\frac{1}{1-\alpha}} > 0$$

From (20), ie: $c^* = (k^*)^{\alpha} - \delta k^*$:

$$c^* = \Big(\frac{\alpha\beta}{1-\beta+\delta\beta}\Big)^{\frac{\alpha}{1-\alpha}} - \delta\Big(\frac{\alpha\beta}{1-\beta+\delta\beta}\Big)^{\frac{1}{1-\alpha}} > 0$$

From (21), ie $\underbrace{\frac{\beta}{1+\theta}\frac{1}{c^*}m^*}_{B(m)} = \underbrace{\frac{1}{c^*}m^* - 1}_{A(m)}$:

$$m^* = \frac{1+\theta}{1+\theta-\beta} \cdot c^* > 0$$

→ unique values $k^* >, c^* > 0$, $m^* > 0$

→ **notice:** no second steady-state solution $m^* = 0$!

IV Stability of steady states

Graphical characterization: example economy 1

Example 1: $\theta \geqslant 0$, $y = k^{\alpha}$, and $v(c) + \phi(m) = \log(c) + \log(m)$

Step II: Phase diagram around $k^* > 0$, $c^* > 0$, $m^* > 0$

Step II involves in itself a 2-step procedure:
IIa) $\rightarrow$ establish (local) saddlepath-stability of the subsystem (25)-(26) in c_t^T and k_t around $k^* > 0$, $c^* > 0$
(*notice:* for this step the particular specifications of $f(k)$ and $v(c) + \phi(m)$ do not matter)

IIb) $\rightarrow$ establish saddlepath-stability of the difference equation in m_t (27) around $m^* > 0$, taken as given $k^* > 0$, $c^* > 0$
(*notice:* for this step the specification of $v(c) + \phi(m)$ as $\log(c) + \log(m)$ matters)

IV Stability of steady states

Graphical characterization: example economy 1

Step IIa): Phase diagram of the subsystem (25)-(26) in c_t^T and k_t
$\longrightarrow$ we need 1st order approximate versions of eqns (25) and (26), with 'appropriate' terms of type Δc_{t+1} and Δk_{t+1}:

- for the **Euler equation** (25) use

$$v_c(c_{t+1}^T) \approx v_c(c_t^T) + v_{cc}(c_t^T) \cdot \underbrace{(c_{t+1}^T - c_t^T)}_{\Delta c_{t+1}^T}$$

to rewrite (25) approximately as

$$\beta(1 + f_k(k_{t+1}) - \delta)[\underbrace{v_c(c_t^T) + v_{cc}(c_t^T) \cdot \Delta c_{t+1}^T}_{\approx v_c(c_{t+1}^T)}]) \approx v_c(c_t^T)$$

$$\Leftrightarrow \quad \Delta c_{t+1}^T \approx -\frac{v_c(c_t^T)}{v_{cc}(c_t^T)} \cdot [1 - \frac{1}{\beta(1 + f_k(k_{t+1}) - \delta)}] \tag{30}$$

IV Stability of steady states

Graphical characterization: example economy 1

Step IIa): Phase diagram of the subsystem in c_t^T and k_t

Dynamic implication of the just established eqn (30), ie

$$\Leftrightarrow \quad \Delta c_{t+1}^T \approx -\frac{v_c(c_t^T)}{v_{cc}(c_t^T)} \cdot [1 - \frac{1}{\beta(1 + f_k(k_{t+1}) - \delta)}]$$

- notice: $-\frac{v_c(c_t^T)}{v_{cc}(c_t^T)} > 0$
- eqn features no dynamics in k, only in c^T
- $\rightarrow$ if $k_{t+1} = k^* \Rightarrow \Delta c_{t+1}^T = 0$ and

$$\Delta c_{t+1}^T \lesseqgtr 0 \;\text{ if }\; k_{t+1} \gtreqless k^*$$

IV Stability of steady states

Graphical characterization: example economy 1

Step IIa): Phase diagram of the subsystem in c_t^T and k_t

- for the **resource constraint** (26), no approximation needed, ie rewrite

$$c_t^T + k_{t+1} = f(k_t) + (1 - \delta)k_t$$

as

$$\Delta k_{t+1} = f(k_t) - \delta k_t - c_t^T \tag{31}$$

Dynamic implication of (31):

- eqn features no dynamics in c^T, only in k
- $\rightarrow$ if $c_t^T = f(k_t) - \delta k_t \Rightarrow \Delta k_{t+1} = 0$ and

$$\Delta k_{t+1} \lesseqgtr 0 \ \text{ if } \ c_t^T \gtreqless f(k_t) - \delta k_t$$

IV Stability of steady states

Graphical characterization: example economy 1

Step IIa): Phase diagram of the subsystem in c_t^T and k_t

$\rightarrow$ Combine the information contained in the two expressions

$$\begin{aligned} \Delta k_{t+1} &\lesseqgtr 0 \text{ if } c_t^T \gtreqless f(k_t) - \delta k_t \\ \Delta c_{t+1}^T &\lesseqgtr 0 \text{ if } k_{t+1} \gtreqless k^* \end{aligned}$$

to represent the dynamics in c_t^T and k_t via a phase diagram:

Here: *Figure 1 (Example 1: Dynamics in c^T and k)*

IV Stability of steady states

Graphical characterization: example economy 1

Step IIa): Comments on the phase diagram of the subsystem in c_t^T and k_t

- Arrows in *Figure 1* indicate regions of stability and instability around $k^* > 0$, $c^* > 0$
- Important information not yet used: (i) $k \geqslant 0$, and (ii) TV-condition (8)
- For any initial departure of the state variable such that $k_{-1} \neq k^*$: **Saddle-path configuration**, i.e. there exists a unique choice of the control variable $c_0 = c_{-1}^T$ such that the economy jumps on the saddlepath and converges over time towards the steady state k^*, c^*
- For all other choices, the dynamics ultimately drift away from k^*, c^*
- Moreover, such choices can be ruled out because the economy would eventually hit
  either: a **'path of rising consumption and falling capital'** on which k would become negative (but this cannot be)
  or: a **'path of falling consumption and rising capital'** on which the present value of lifetime consumption would become smaller than the present value of lifetime income (but this cannot be optimal)

IV Stability of steady states

Graphical characterization: example economy 1

Step IIb): Phase diagram of m_t around $m^* > 0$, taken as given $k^* > 0,\ c^* > 0$

$\rightarrow$ using $\nu(c) + \phi(m) = \log(c) + \log(m)$, (27) becomes:

$$B(m_{t+1}) \equiv \frac{\beta}{1+\theta}\frac{1}{c^*} \cdot m_{t+1} = \frac{1}{c^*}m_t - 1 \equiv A(m_t) \tag{32}$$

$$\Leftrightarrow \quad m_{t+1} = \underbrace{\frac{1+\theta}{\beta}}_{a_m > 1} m_t - \frac{1+\theta}{\beta}c^* \tag{33}$$

$\rightarrow$ no linearization needed,ie
dynamics in m_t governed by a linear first-order difference equation

IV Stability of steady states

Graphical characterization: example economy 1

Step IIb): Phase diagram of m_t **around** $m^* > 0$, **taken as given** $k^* > 0$, $c^* > 0$

$\rightarrow$ to represent the dynamics of (32) in m_t via a phase diagram, use

$$\frac{\beta}{1+\theta}\frac{1}{c^*} < \frac{1}{c^*},$$

ie the slope coefficient of $B(m_{t+1})$ is smaller than the one of $A(m_t)$:

Here: *Figure 2 (Example 1: Dynamics in m)*

IV Stability of steady states

Graphical characterization: example economy 1

Step IIb): Comments on the phase diagram of the dynamics in m_t

- Arbitrary initial values of type m_0' or m_0'' in *Figure 2* lead to unstable dynamics, moving away from m^*.
  This reflects that (33) is for arbitrary initial values an unstable difference equation (in the backwardlooking sense).
- But the backwardlooking perspective is misleading since the sequence m_t has no initial condition, ie if m_0 jumps directly to the unique value m^* dynamics are stable (and the absence of transitory dynamics is a special case of forward-looking **saddlepath-stability**)
- Moreover, $m_0 = m^*$ is optimal, since:
  if $m_0' < m^*$, m_T becomes negative for some finite horizon T (but this cannot be) and
  if $m_0'' > m^*$, m_t grows at the rate $\frac{1+\theta}{\beta}$. However, the TV-condition (8) requires

$$\lim_{T\to\infty} \beta^T \cdot v_c(c^*) \cdot m_T = 0$$

  and $\theta \geqslant 0$ implies that this condition will be violated (but this cannot be)

IV Stability of steady states

Example economy 1

Interpretation and comments:

- In terms of **economic insights**, the particular specification of additively separable preferences used in Example 1 illustrates that the basic MIU model has the potential to extend superneutrality to transitory dynamics, ie the specification supports the notion that 'money can act as a veil' in the strongest possible sense
- In terms of its **technical features**, example 1 exhibits a unique steady state with (locally) saddlepath stable dynamics, ie by combining the restrictions from both initial and terminal conditions the dynamics of all variables are stable and uniquely defined around this steady state
- This concept is a standard one which is routinely used in macro-models with forward-looking agents
- In **stochastic extensions** of models of this type it implies that small shocks (within the neighbourhood around a steady state) trigger stable and predictable reactions of optimizing agents such that the economy eventually returns to the starting point

IV Stability of steady states

Example economy 1

Interpretation and comments:

- In **large-scale macro models** (used for forecasts and policy simulations), which, in any case, are not recursive, this configuration cannot be verified in simple phase diagrams. Instead, these models need to be solved numerically. Yet, the basic intuition for the possibility of saddlepath–stable dynamics of such systems is in line with example 1
- **Criticism:** for saddlepath-stable configurations, the role of the 'fundamentals of the economy' (here captured by the single value k_{-1}) is very strong (and for many applications too strong)
- **Alternative view:**
  $\longrightarrow$ Models should allow for **self-fulfilling fluctuations**, driven by non-fundamental 'animal spirits' (Keynes).
  $\longrightarrow$ With equally simple model ingredients, this can be achieved if the dynamics implied by the system of difference equations are somewhat different, leading to locally **indeterminate (but still stable) dynamics** (and we will briefly return to this when we sketch the analytics of stability issues below)
  $\longrightarrow$ More far-reaching criticism: **rational expectations assumption as such to be modified** (eg via learning) or entirely abandoned

IV Stability of steady states

Graphical characterization: example economy 2

Example 2: $\theta \geqslant 0$, $y = k^{\alpha}$, and $v(c) + \phi(m) = \log(c) + \frac{1}{1-\sigma} m^{1-\sigma}$, $\sigma \in (0,1)$

Step I: steady state calculation

From (19), (20): values of k^* and c^* identical with those of example 1, ie:

$$k^* = \left(\frac{\alpha\beta}{1-\beta+\delta\beta}\right)^{\frac{1}{1-\alpha}} > 0 \text{ and } c^* = \left(\frac{\alpha\beta}{1-\beta+\delta\beta}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{\alpha\beta}{1-\beta+\delta\beta}\right)^{\frac{1}{1-\alpha}} > 0$$

From (21), ie $\underbrace{\frac{\beta}{1+\theta}\frac{1}{c^*}m^*}_{B(m)} = \underbrace{\frac{1}{c^*}m^* - (m^*)^{1-\sigma}}_{A(m)}$:

$$\begin{aligned} m_1^* &= \left(\frac{1+\theta}{1+\theta-\beta} \cdot c^*\right)^{\frac{1}{\sigma}} > 0 \\ m_2^* &= 0 \end{aligned}$$

→ unique positive values $k^* >, c^* > 0$, $m_1^* > 0$

→ **but:** existence of a 2nd solution $m_2^* = 0$!

IV Stability of steady states

Graphical characterization: example economy 2

Example 2: $\theta \geqslant 0$, $y = k^{\alpha}$, and $\nu(c) + \phi(m) = \log(c) + \frac{1}{1-\sigma} m^{1-\sigma}$, $\sigma \in (0, 1)$

Step II: Phase diagram around $k^* >, c^* > 0$, $m_1^* > 0$

Step II, again, involves in itself a 2-step procedure:
IIa) $\rightarrow$ **identical to example** 1, ie (local) saddlepath-stability of the subsystem (25)-(26) in c_t^T and k_t around $k^* > 0$, $c^* > 0$
(*remember:* for this step the particular specifications of $f(k)$ and $\nu(c) + \phi(m)$ do not matter)

IIb) $\rightarrow$ **saddlepath-stability** of the difference equation in m_t (27) around $m_1^* > 0$, taken as given $k^* > 0$, $c^* > 0$, **vanishes** since dynamics may converge against $m_2^* = 0$
(*notice:* for this step the specification of $\nu(c) + \phi(m)$ as $\log(c) + \frac{1}{1-\sigma} m^{1-\sigma}$, $\sigma \in (0, 1)$ matters)

IV Stability of steady states

Graphical characterization: example economy 2

Step IIb): Phase diagram of m_t around $m_1^* > 0$, for given $k^* > 0$, $c^* > 0$

$\rightarrow$ Using $v(c) + \phi(m) = \log(c) + \frac{1}{1-\sigma} m^{1-\sigma}$, (27) becomes

$$B(m_{t+1}) \equiv \frac{\beta}{1+\theta} \frac{1}{c^*} \cdot m_{t+1} = \frac{1}{c^*} m_t - \underbrace{m_t^{1-\sigma}}_{\phi_{m_t}(m_t) \cdot m_t} \equiv A(m_t) \qquad (34)$$

$\rightarrow$ According to (34), dynamics governed by a **non-linear** first-order difference equation in m_t

$\rightarrow$ **Linearized version** of (34) around $m_1^* = (\frac{1+\theta}{1+\theta-\beta} \cdot c^*)^{\frac{1}{\sigma}} > 0$ (where only the term $\phi_{m_t}(m_t) \cdot m_t$ on the RHS of (34) requires linearization)

$$\frac{\beta}{1+\theta} \frac{1}{c^*} \cdot (m_{t+1} - m_1^*) = [\frac{1}{c^*} - (1-\sigma)(m_1^*)^{-\sigma}](m_t - m_1^*)$$

$$\Leftrightarrow \quad m_{t+1} - m_1^* = \underbrace{[\sigma \frac{1+\theta}{\beta} + 1 - \sigma]}_{a_m > 1 \text{ for } \forall \ \sigma \in (0,1)} \cdot (m_t - m_1^*) \qquad (35)$$

IV Stability of steady states

Graphical characterization: example economy 2

Step IIb): Phase diagram of m_t **around** $m_1^* > 0$, **taken as given** $k^* > 0$, $c^* > 0$

→ represent the **dynamics** of the original, **non-linearized equation** (34) in m_t via a phase diagram:

Here: *Figure 3 (Example 2: Dynamics in m)*

IV Stability of steady states

Graphical characterization: example economy 2

Step IIb): Comments on the phase diagram of the dynamics in m_t

- Complete (ie non-linear) configuration is much richer than the linearized dynamics around m_1^*
- Again, for arbitrary initial values of $m_0 \neq m_1^*$ dynamics are unstable
- $\rightarrow$ if $m_0'' > m_1^*$:
  all paths to be ruled out by violations of the TV-condition (see ex. 1)
- if $m_0' < m_1^*$:
  $\rightarrow$ in general, also to be ruled out: m_T will become negative for large T
  $\rightarrow$ yet: for some value $m_0' < m^*$ dynamics converge against $m_2^* = 0$
  $\rightarrow$ specifically: if the system hits $\widetilde{m}$ it moves in the next period to $m_2^* = 0$
  $\rightarrow$ this requires an infinite jump in the price level ('hyperinflation')
  $\rightarrow$ and then the system stays at $m_2^* = 0$ forever

IV Stability of steady states

Graphical characterization: example economy 2

Step IIb): Comments on the phase diagram of the dynamics in m_t

- **Important:** dynamics towards $m_2^* = 0$ do not violate the optimality conditions derived from forwardlooking behaviour. Why?
  $\longrightarrow$ At $\widetilde{m}$ to be satisfied:

  $$\phi_m(\widetilde{m}) = \nu_c(c^*)$$

  $\longrightarrow$ Compare this with the first-order condition:

  $$\phi_m(m_t) = \frac{i_t}{1+i_t} \cdot \nu_c(c^*) = \frac{1}{1+\frac{1}{i_t}} \cdot \nu_c(c^*)$$

  $\longrightarrow$ Use $i_t = (1+r^*) \cdot \frac{P_{t+1}}{P_t} - 1$. Hence, for given P_t, $i_t \longrightarrow \infty$ as $P^e_{t+1} \longrightarrow \infty$ ('rationally expected hyperinflation'), implying $\frac{i_t}{1+i_t} \longrightarrow 1$ such that $\phi_m(\widetilde{m}) = \nu_c(c^*)$ can be rationalized

IV Stability of steady states

Graphical characterization: example economy 2

Step IIb): Comments on the phase diagram of the dynamics in m_t

- **Technically**, what is the difference between the 2 examples?
  $\rightarrow$ in Example 1: $\lim_{m \rightarrow 0} \phi(m) \rightarrow -\infty$, while in Example 2: $\lim_{m \rightarrow 0} \phi(m) = 0$
  $\rightarrow$ To rule out the possibility of hyperinflationary dynamics (ie Ex. 1), money must be so necessary that the utility loss is sufficiently large (ie infinite!) if real balances go to zero

IV Stability of steady states

Example economy 2

Interpretation and comments:

In terms of its **technical features**, example 2 illustrates some important insights

- The **linearization** of macroeconomic models, while often inevitable, can come at a significant cost since the **'global'** behaviour of economies can be very different from predictions obtained from **'local'** characterizations: $\rightarrow$ in our case: the possibility of hyperinflationary dynamics would not have been captured if we had used the linear equation (34) instead of the original non-linear one (35)
- The existence of **multiple steady states** leads to global coordination problems and questions of equilibrium selection
- These issues are at odds with the strong uniqueness property of saddlepath-stable solutions

IV Stability of steady states

Example economy 2

Interpretation and comments:

In terms of **economic insights**, example 2 has a number of interesting and partly controversial features:

- The possibility of a purely speculative hyperinflation (where for $\theta \geqslant 0$ real balances m_t ultimately go to zero, ie π_t rises faster than θ, leading to a complete collapse of the monetary equilibrium) is the flip side of the complete dichotomy between the nominal and real side of the model
- Neutrality and superneutrality facilitate the possibility of a self-fulfilling and 'de-coupled' hyperinflation which does not affect the real side of the economy

→ How **plausible** is this? Why should it better be seen as a 'degenerate' story?

- The qualification as a 'degenerate' scenario does **not** refer per se to the particular functional choice of $v(c) + \varphi(m) = \log(c) + \frac{1}{1-\sigma} m^{1-\sigma}$
- It rather refers to a well-understood **fragility of the model** itself

→ To rule out the hyperinflationary scenario not much is needed: as long as the central bank stands ready to guarantee some **minimal real redemption value for money**, non-fundamental **hyperinflationary dynamics**, by backward-induction, **can never take off**

IV Stability of steady states

Example economy 2

Interpretation and comments:

$\longrightarrow$ in reality, such qualifications of pure fiat money regimes exist, ie central banks hold reserves like gold and implement their standing operations by investing in different types of assets
$\longrightarrow$ interesting different traditions of monetary policy implementation:

- **US:** tradition of 'treasuries only' (outright purchases); recently extended to various private paper facilities
- **Eurosystem:** tradition of accepting government and private paper as collateral; recently extended to outright purchases of (some) gov't paper
- **in either tradition:** recognition of (crisis-related) lender of last resort function of central banks to stem financial panics (via discount window)

V Stability of steady states: analytical solution

Analytical characterization of the (in)stability of linearized systems:
→ Reconsider the above established linearized system (28)-(29),ie:

$$\begin{bmatrix} c_{t+1}^T - c^* \\ k_{t+1} - k^* \end{bmatrix} = A \cdot \begin{bmatrix} c_t^T - c^* \\ k_t - k^* \end{bmatrix}$$

- -

$$m_{t+1} - m^* = a_m \cdot (m_t - m^*)\,,$$

→ where $A = \begin{bmatrix} a_{11}(k^*, c^*) & a_{12}(k^*, c^*) \\ a_{21}(k^*, c^*) & a_{22}(k^*, c^*) \end{bmatrix}$ is a 2x2-matrix and $a_m = a_m(k^*, c^*, m^*)$ is a scalar

Aim:
→ i) Derive **analytically** the saddlepath-stable solution of the **linearized dynamics** around (k^*, c^*, m^*)
→ ii) Extend the reasoning to a **general classification of stability patterns of linear systems** where A is a nxn-matrix and we have n_1 predetermined and $n_2 = n - n_1$ forwardlooking variables

V Stability of steady states: analytical solution

Analytical characterization of the (in)stability of linearized systems:

$\rightarrow$ The (in)stability of linearized systems of difference equations is determined by their characteristic roots or, equivalently, their eigenvalues, denoted by λ

$\rightarrow$ A 3x3-system has generically 3 distinct eigenvalues (and, for simplicity, we consider $|\lambda_i| \neq 1$)

$\rightarrow$ Special constellation of (28)-(29): because of the independence of (29), the dynamics in m_t are governed by $\lambda_3 = a_m$, while λ_1 and λ_2 are linked to the 2x2-matrix A

V Stability of steady states: analytical solution

Analytical characterization of the (in)stability of linearized systems:
Consider first:

$$m_{t+1} - m^* = \underbrace{a_m}_{\lambda_3} \cdot (m_t - m^*)$$

$\rightarrow$ The eigenvalue a_m induces a linear mapping such that the scalar argument $(m_t - m^*)$ is scaled up or down over time, depending on whether $|a_m| \gtrless 1$

Backwardlooking interpretation:
If $|\lambda_3| < 1$: stability for arbitrary initial conditions $m_t \neq m^*$

Forwardlooking interpretation (see Ex 1 and 2):
$\rightarrow$ Since m_t introduced as a forwardlooking variable w/o initial (but with terminal) condition stability requires $|\lambda_3| > 1$
$\rightarrow$ Why? Rewrite the eqn as

$$m_t - m^* = \frac{1}{\lambda_3}(m_{t+1} - m^*) = (\frac{1}{\lambda_3})^T \cdot (m_{t+T} - m^*),$$

implying $m_t = m^*$ since the term $m_{t+T} - m^*$ is bounded by the terminal condition such that $\lim_{T \to \infty} (\frac{1}{\lambda_3})^T \cdot (m_{t+T} - m^*) = 0$

V Stability of steady states: analytical solution

Analytical characterization of the (in)stability of linearized systems:

Consider now:

$$\begin{bmatrix} c_{t+1}^T - c^* \\ k_{t+1} - k^* \end{bmatrix} = A \cdot \begin{bmatrix} c_t^T - c^* \\ k_t - k^* \end{bmatrix}$$

$\rightarrow$ Is there a counterpart to the just discussed scalar $a_m = \lambda_3$ for the 2x2-system governed by A?

$\rightarrow$ To simplify notation let $h_{t+1} = A \cdot h_t$ with: $h_t \equiv \begin{bmatrix} c_t^T - c^* \\ k_t - k^* \end{bmatrix}$

$\rightarrow$ **Special case:** Assume

$$A \cdot h_t = \lambda \cdot h_t = h_{t+1},$$

ie the matrix A induces a linear mapping such that the vector argument h_t is scaled up or down over time, depending on whether $|\lambda| \gtrless 1$

In such special case denotes:

i) the scalar λ an **eigenvalue** of the matrix A

ii) the vector $h \equiv q$ an **eigenvector** of A, associated with the eigenvalue λ

V Stability of steady states: analytical solution

Analytical characterization of the (in)stability of linearized systems:

→ From the eqn

$$A \cdot q = \lambda \cdot q$$

eigenvalues solve the equation

$$[A - \lambda I] \cdot q = 0, \quad \text{with:} \;\; I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

→ For non-trivial solutions (ie $q \neq 0$), the matrix $[A - \lambda I]$ needs to be 'singular' (ie the inverse of $[A - \lambda I]$ does not exist), leading to the so-called **characteristic equation**:

$$|A - \lambda I| = 0 \quad \Leftrightarrow \quad \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

Equivalently, the characteristic equation can be written as

$$\lambda^2 - \underbrace{(a_{11} + a_{22})}_{Tr(A)}\lambda + \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{Det(A)} = 0 \tag{36}$$

V Stability of steady states: analytical solution

Analytical characterization of the (in)stability of linearized systems:

$\rightarrow$ The characteristic eqn (36) is a quadratic eqn in λ

$\rightarrow$ There exist generically two different eigenvalues λ_1 and λ_2, ie

$$\lambda_{1,2} = \frac{1}{2} \cdot Tr(A) \pm \frac{1}{2} \cdot \sqrt{(Tr(A))^2 - 4 \cdot Det(A)}$$

$\rightarrow$ with associated eigenvectors $q_1 = \begin{pmatrix} \mu_1 \\ \overline{q}_1 \cdot \mu_1 \end{pmatrix}$ and $q_2 = \begin{pmatrix} \mu_2 \\ \overline{q}_2 \cdot \mu_2 \end{pmatrix}$

$\rightarrow$ since each λ_i generates 2 linearly dependent equations, the associated eigenvectors have a unique direction (via $\overline{q}_i$), but not a particular length

Some simplifying **notation**:

$\rightarrow$ 2x2$-$**Matrix** Q **of stacked eigenvectors:**

$$Q = [q_1 \ q_2] = \begin{bmatrix} \mu_1 & \mu_2 \\ \overline{q}_1 \cdot \mu_1 & \overline{q}_2 \cdot \mu_2 \end{bmatrix}$$

$\rightarrow$ 2x2$-$**Diagonal matrix** Λ **of eigenvalues:**

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

V Stability of steady states: analytical solution

Analytical characterization of the (in)stability of linearized systems:

→ Write the definition of eigenvalues and eigenvectors in matrix form:

$$A \cdot Q = A \cdot [q_1 \;\, q_2] = [q_1 \;\, q_2] \cdot \left[\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array} \right] = Q \cdot \Lambda$$

→ Since $Q \cdot Q^{-1} = I$, rewrite the matrix A via its **'Jordan canonical form'**:

$$A = Q \cdot \Lambda \cdot Q^{-1},$$

where it is customary to order the eigenvalues in Λ by size (starting with the smallest one in the top left corner of Λ)

→ The **inverse matrix** Q^{-1} of Q is also 2x2-matrix**:**

$$Q^{-1} = \frac{1}{Det(Q)} \left[\begin{array}{cc} \overline{q}_2 \cdot \mu_2 & -\mu_2 \\ -\overline{q}_1 \cdot \mu_1 & \mu_1 \end{array} \right] \equiv \left[\begin{array}{cc} \widetilde{q_{11}} & \widetilde{q_{12}} \\ \widetilde{q_{21}} & \widetilde{q_{22}} \end{array} \right]$$

V Stability of steady states: analytical solution

Analytical characterization of the (in)stability of linearized systems:

$\longrightarrow$ Define a **new vector** z_t containing linear combinations of the initial variables with weights taken from Q^{-1} such that

$$z_t = \begin{pmatrix} z_{1,t} \\ z_{2,t} \end{pmatrix} = Q^{-1} \cdot h_t,$$

ie

$$z_{1,t} = \widetilde{q_{11}} \cdot h_{1,t} + \widetilde{q_{12}} \cdot h_{2,t} \text{ and } z_{2,t} = \widetilde{q_{21}} \cdot h_{1,t} + \widetilde{q_{22}} \cdot h_{2,t}$$

$\longrightarrow$ Rewrite the initial $2x2-$system (28), ie

$$h_{t+1} = A \cdot h_t,$$

using $A = Q \cdot \Lambda \cdot Q^{-1}$ as

$$Q^{-1} \cdot h_{t+1} = z_{t+1} = \Lambda \cdot z_t \tag{37}$$

Notice: Since Λ is a diagonal matrix, eqn (37) consists of two 'de-coupled' first-order difference eqns, qualitatively similar to (29), ie we can write it as

$$\begin{aligned} z_{1,t+1} &= \lambda_1 \cdot z_{1,t} \\ z_{2,t+1} &= \lambda_2 \cdot z_{2,t} \end{aligned}$$

V Stability of steady states: analytical solution

Analytical characterization of the (in)stability of linearized systems:

→ The pair of equations

$$z_{1,t+1} = \lambda_1 \cdot z_{1,t} \quad \text{and} \quad z_{2,t+1} = \lambda_2 \cdot z_{2,t} \tag{38}$$

describe the **general solution** of the $2x2-$system

$$h_{t+1} = A \cdot h_t$$

→ **Equivalently**, the general solution can be written as

$$h_t = \begin{pmatrix} h_{1,t} \\ h_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \overline{q}_1 \cdot \mu_1 \end{pmatrix} \cdot \lambda_1^t + \begin{pmatrix} \mu_2 \\ \overline{q}_2 \cdot \mu_2 \end{pmatrix} \cdot \lambda_2^t \tag{39}$$

→ Using either (38) or (39), the **definite solution** can be obtained if one uses the initial and terminal conditions

V Stability of steady states: analytical solution

Analytical characterization of the (in)stability of linearized systems:
→ **Recall:** one predetermined variable (k) and one forwardlooking variable (c)
→ **Assume:** $|\lambda_1| < 1$ and $|\lambda_2| > 1$

Notice: it can be verified that the matrix A derived from the linearized eqns (25) and (26) generically satisfies this pattern of eigenvalues

Since $|\lambda_2| > 1$ solve the second eqn forward, ie rewrite it as $z_{2,t+1} = \lambda_2 \cdot z_{2,t}$

$$z_{2,t} = \frac{1}{\lambda_2} \cdot z_{2,t+1} = \left(\frac{1}{\lambda_2}\right)^T \cdot z_{2,t+T}$$

and deduce from $\lim_{T\to\infty} \left(\frac{1}{\lambda_2}\right)^T \cdot z_{2,t+T} = 0$ the solution

$$z_{2,t} = \widetilde{q_{21}} \cdot \underbrace{h_{1,t}}_{c_t^T - c^*} + \widetilde{q_{22}} \cdot \underbrace{h_{2,t}}_{k_t - k^*} = 0,$$

implying that the **forwardlooking** (control) **variable** c should be set s.t.

$$c_t^T - c^* = -\frac{\widetilde{q_{22}}}{\widetilde{q_{21}}} \cdot (k_t - k^*) \qquad (40)$$

V Stability of steady states: analytical solution

Analytical characterization of the (in)stability of linearized systems:

→ What about the dynamics in $(k_t - k^*)$?
→ Use the first eqn

$$z_{1,t+1} = \lambda_1 \cdot z_{1,t} \quad \text{with:} \quad z_{1,t} = \widetilde{q_{11}} \cdot h_{1,t} + \widetilde{q_{12}} \cdot h_{2,t}$$

→ Substitute eqn (40),ie

$$\underbrace{c_t^T - c^*}_{h_{1,t}} = -\frac{\widetilde{q_{22}}}{\widetilde{q_{21}}} \cdot \underbrace{(k_t - k^*)}_{h_{2,t}}.$$

in the first eqn to obtain

$$[\widetilde{q_{12}} - \widetilde{q_{11}} \frac{\widetilde{q_{22}}}{\widetilde{q_{21}}}] \cdot (k_{t+1} - k^*) = \lambda_1 \cdot [\widetilde{q_{12}} - \widetilde{q_{11}} \frac{\widetilde{q_{22}}}{\widetilde{q_{21}}}] \cdot (k_t - k^*),$$

implying for the **law of motion** of the **state variable** k :

$$k_{t+1} - k^* = \lambda_1 \cdot (k_t - k^*) \tag{41}$$

V Stability of steady states: analytical solution

Comments on the solution and generalizations

Solution:

→ The two eqns (40) and (41), ie

$$\begin{aligned} k_{t+1} - k^* &= \lambda_1 \cdot (k_t - k^*) \\ c_t^T - c^* &= c_{t+1} - c^* = -\frac{\widetilde{q_{22}}}{\widetilde{q_{21}}} \cdot (k_t - k^*) \end{aligned}$$

are the solutions, summarizing $\forall t \geqslant -1$ the behaviour of the linearized versions of (25) and (26), as captured by the matrix A, along the linear saddlepath until convergence of k_t and c_t^T against k^* and c^*

→ The derivation of (40) and (41) has used that we have 1 stable and 1 unstable eigenvalue which we have matched with the single initial and the single terminal condition

V Stability of steady states: analytical solution

Comments on the solution and generalizations

Initializing the system at $t = -1$:

$\longrightarrow$ Recall: k_{-1} is the single initial condition of the system (40) and (41)
$\longrightarrow$ Consider the two eqns at $t = -1$, ie

$$\begin{aligned} k_0 - k^* &= \lambda_1 \cdot (k_{-1} - k^*) \\ c_{-1}^T - c^* &= c_0 - c^* = -\frac{\widetilde{q_{22}}}{\widetilde{q_{21}}} \cdot (k_{-1} - k^*), \end{aligned}$$

implying that we managed to initialize the law of motion for k_t and c_t by the single initial condition k_{-1}
$\longrightarrow$ for all $t > -1$: unique values of k_t and c_t determined recursively by (40) and (41)

V Stability of steady states: analytical solution

Comments on the solution and generalizations

Cross-equation restriction:

- Equations of type (40), ie

$$c_t^T - c^* = -\frac{\widetilde{q_{22}}}{\widetilde{q_{21}}} \cdot (k_t - k^*)$$

  are examples of **cross equation restrictions**
- In general, restrictions of this type, going back to Lucas (1976), are a key feature of macro-models which incorporate forwardlooking behaviour and are intimately linked to the so-called **Lucas critique**
- This critique revolutionized macroeconomic analysis 40 years ago
- The Lucas critique says that econometricians who want to estimate a relationship like (40) need to be aware that coefficients like $-\widetilde{q_{22}} / \widetilde{q_{21}}$ consist not only of **structural ('deep') parameters** like α, β or δ, but also of **policy parameters** (like θ)
- In particular, changes in parameters of policy rules do affect such coefficients, implying that **policy advice based on past estimates of such coefficients will be systematically wrong**

V Stability of steady states: analytical solution

Comments on the solution and generalizations

Cross-equation restriction (cont'd):

- **Remark:** for the **special system** characterized by **additively separable preferences** the single policy parameter θ does not enter the dynamics governed by A, ie for this very special system the Lucas critique does not apply
- However, **in general**, assuming **non-separable preferences** with $u = u(c, m)$ such that one obtains a fully integrated $3x3-$system in k_t, c_t and m_t, the Lucas critique does apply. In other words, the coefficient linking consumption and capital (and, hence, output) will be a function of the policy parameter θ
- In case policymakers announce a systematic change in their policy rule (*here:* 'change in θ'), forwardlooking agents will incorporate this in their decisions. Policy-advice not internalizing this reaction will be misleading

V Stability of steady states: analytical solution

Comments on the solution and generalizations

Generalization I (Large-scale deterministic linear systems):

$\longrightarrow$ Consider an economy characterized by n_1 **predetermined (or state) variables with initial conditions** and $n_2 = n - n_1$ **forwardlooking (or control) variables with terminal conditions**

$$h_{t+1} = \begin{bmatrix} h^P_{t+1} \\ h^F_{t+1} \end{bmatrix} = A \cdot \begin{bmatrix} h^P_t \\ h^F_t \end{bmatrix} = A \cdot h_t,$$

where A is a $nxn-$matrix, h is a $nx1-$vector and h^P and h^F are n_1x1 and n_2x1-vectors of predetermined and forwardlooking variables, respectively

V Stability of steady states: analytical solution

Comments on the solution and generalizations

Generalization I (Large-scale deterministic linear systems):

Blanchard-Kahn (1980) conditions:

- If the system is to have a **unique stationary equilibrium**, n_1 eigenvalues of the matrix A need to satisfy $|\lambda_i| < 1$, $i = 1, 2, .., n_1$, while n_2 eigenvalues need to satisfy $|\lambda_j| > 1$, $j = n_1 + 1, .., n$.
- If there are fewer than n_2 eigenvalues with $|\lambda_j| > 1$, then the system is characterized by **multiple stationary equilibria (indeterminacy)**
- If there are more than n_2 eigenvalues with $|\lambda_j| > 1$, then **no solution exists**

- If a **unique stationary equilibrium** exists, the solution takes the form:

$$h_{t+1}^P = M \cdot h_t^P \quad \text{and} \quad h_t^F = C \cdot h_t^P$$

- If there exist **multiple stationary equilibria (indeterminacy)**:
  $\rightarrow$ possibility of **self-fulfilling fluctuations ('animal spirits')**

V Stability of steady states: analytical solution

Comments on the solution and generalizations

Comment 1: Unit roots

- If eigenvalues satisfy the borderline case of $|\lambda_i| = 1$ ('unit root'), the classification can be adjusted:
  If the system is to have a **unique equilibrium**, n_1 eigenvalues of the matrix A need to satisfy $|\lambda_i| \leq 1$, $i = 1, 2, .., n_1$, while n_2 eigenvalues need to satisfy $|\lambda_j| > 1$, $j = n_1 + 1, .., n$.
- **Intuition:** Eigenvalues satisfying $|\lambda_i| = 1$ create special dynamics in the sense that the system will not return to its starting point, but neither will it explode
- **Numerically**, such constellation is not generic (ie the probability that we hit such special value for 'arbitrary' matrices A is zero)
- However, many models have deliberately a **theoretical** design such that unit roots do matter (eg permanent as opposed to transitory technology or taste shocks etc)

V Stability of steady states: analytical solution

Comments on the solution and generalizations

Comment 2: Level changes vs. percentage deviations

- Typically, to make reactions between the various variables comparable, the representative entries of h_t^P and h_t^F are specified as **percentage deviation** of some variable from its steady state, like, eg,

$$h_i^P = \widehat{k}_t = \frac{k_t - k^*}{k^*} \quad \text{or} \quad h_j^F = \widehat{c}_t = \frac{c_t - c^*}{c^*},$$

  and not the absolute differences (as done above)
- Variables with a **hat-notation** ($\widehat{k}_t$, $\widehat{c}_t$ etc.) typically describe such percentage deviation
- This change in representation matters only at the stage when the linearizations are done, but not afterwards