Part II
Money and Public Finance
Lecture 6
Selected Issues from a Normative Perspective

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Monetary and Fiscal Policy Issues in General Equilibrium
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Motivation

What about **optimal policy** if both monetary and fiscal policies are associated with distortions?

$\rightarrow$ Should the nominal interest rate be optimally set such that $i = 0$?
$\rightarrow$ Interesting debate between Friedman (1969) and Phelps (1973) with two competing conjectures:

Friedman: Remove inefficiencies by equating the social marginal cost of producing money and the private marginal cost of holding money by setting $i = 0$

Phelps: Assume regular taxes are distortionary: "...*If, as is often maintained, the demand for money is highly interest-inelastic, then liquidity is an attractive candidate for heavy taxation at least from the standpoint of monetary and fiscal efficiency*" (p.82)

Motivation

Phelps or **Friedman: "Who is right?"**

More appropriate question: What assumptions need to be satisfied for Friedman or Phelps to be right?

To understand this:
- a **general equilibrium perspective** is needed such that we can identify the key mechanisms behind the two conjectures
- to study optimality: we will use basic **concepts for optimal taxation from public finance**

Some Taxonomy from Public Finance: Ramsey theory

→ 2 (related) ways to solve the Ramsey problem of optimal policymaking

I) Direct (or dual) approach
→ Optimization via optimal choice of policy instruments, based on indirect utility function of representative private HH
→ Intuition straightforward, but in practice solutions often tricky

II) Indirect (or primal) approach
→ Optimization via optimal choice of allocation, subject to certain constraints
→ Intuition initially maybe less straightforward, but solutions can often be relatively easily obtained

Some Taxonomy from Public Finance: Ramsey theory

I) Direct (or dual) approach

Ingredients:

- **Budget-feasible policy** denoted by π
(where π typically can be thought of as a combination of monetary and fiscal policies s.t. $\pi = (\tau, R)$, with $\tau = (\tau^c, \tau^n)$ etc.)

- **Private sector equilibrium allocation** x, associated with any given policy π, such that $x(\pi)$

- **Ramsey problem**: max $U(x(\pi))$ w.r.t. π

- **Ramsey equilibrium**: describes the best private sector equilibrium and consists of the optimal policy π^* (subject to budget-feasibility) and the welfare maximizing private sector allocation $x^* = x(\pi^*)$ induced by π^*

Some Taxonomy from Public Finance: Ramsey theory

II) Indirect (or primal) approach

Alternative strategy for solving the same problem by setting up the Ramsey allocation problem

Ingredients:

- max $U(x)$ subject to the constraint that the optimal allocation x^* can be implemented as a competitive equilibrium via a budget-feasible policy, with $\pi^* = x^{-1}(x^*)$

- **Implementability constraint:** summarizes restrictions on achievable allocations

A toy model to fix ideas of Ramsey optimality
Direct approach

→ To see how these concepts work (and why they are equivalent when correctly set up), consider a simply toy model that can be solved by hand under the direct (dual) and the indirect (primal) approach

Step 1: Household problem:

$$\max_{c,n} \; u(c,n)$$

subject to the private sector budget constraint

$$c = (1-\tau)wn \qquad \text{with: } \tau \in (0,1) \tag{1}$$

(c=consumption, n=labour, w=real wage, τ=labour tax)

→ Private sector allocation rules: use FOC of HH problem

$$(1-\tau)wu_c(c,n) = \underbrace{-u_n(c,\underbrace{n}_{1-l})}_{=\frac{\partial u}{\partial l}} \tag{2}$$

to find $n(\tau)$ and $c(\tau)$

A toy model to fix ideas of Ramsey optimality
Direct approach

Step 2: Ramsey problem:

$$\max_{\tau} \; u(c(\tau), n(\tau))$$

subject to the government budget constraint (assuming g is constant)

$$g = \tau w n(\tau) \tag{3}$$

$\rightarrow$ *Notice:* Step 2 amounts to maximize the indirect utility function w.r.t. τ, after substitution of the demand functions $n(\tau)$ and $c(\tau)$ into $u(c, n)$

Ramsey equilibrium:
- optimal allocations $n^*(\tau^*)$ and $c^*(\tau^*)$, given policy τ^*
- optimal policy choice τ^* which satisfies budget constraint of the government and induces the best private sector allocation (n^*, c^*)

A toy model to fix ideas of Ramsey optimality
Direct approach

General equilibrium closure:

- The production function is linear in labour

$$y = n,$$

with the marginal product of labour normalized to unity such that $w = 1$

- This can be used to verify that the resource constraint of the economy

$$c + g = n$$

can be reproduced from the assumed budget constraints of the private sector and the government, ie

$$c = (1 - \tau)wn \quad \text{and} \quad g = \tau wn,$$

when taken together

A toy model to fix ideas of Ramsey optimality

Direct approach: parametric example

Parametric example:

$$\max_{c,n} \; u(c,n) = c - \frac{1}{2}n^2 \quad s.t. \;\; c = (1-\tau)n$$

Private sector allocation rules (from HH's FOC and priv. sector bc):

$$\begin{aligned} n(\tau) &= 1-\tau \\ c(\tau) &= (1-\tau)\cdot n(\tau) = n(\tau)^2 \end{aligned}$$

Ramsey problem:

$$\max_{\tau} \; \frac{1}{2}(1-\tau)^2 \;\; s.t. \;\; g = \tau n(\tau) = (1-\tau)\tau$$

A toy model to fix ideas of Ramsey optimality
Direct approach: parametric example

Ramsey problem:

$$\max_{\tau} \ \frac{1}{2}(1-\tau)^2 \quad s.t. \ \ g = (1-\tau)\tau$$

Solution is easy: $\rightarrow$ (i) graphically, (ii) analytically

Government budget constraint:

- revenues are quadratic in τ with a Laffer-shape (with max at $\tau = \frac{1}{2}$)
- for given value of g, $0 < g < \frac{1}{4}$: gov't bc solved by two values τ_1 and τ_2, with: $0 < \tau_1 < \tau_2 < 1$

The **indirect utility function** $u(c(\tau), n(\tau)) = \frac{1}{2}(1-\tau)^2$ falls monotonically in τ, for $\tau \in (0,1)$. Thus:

$$\tau^* = \tau_1 = \frac{1}{2} - (\frac{1}{4} - g)^{\frac{1}{2}} = \frac{1}{2} - \frac{1}{2}(1-4g)^{\frac{1}{2}}$$

$$\Rightarrow \quad n^*(\tau^*) = 1 - \tau^* \ \text{ and } \ c^*(\tau^*) = (1-\tau^*)^2$$

A toy model to fix ideas of Ramsey optimality
Indirect approach

Alternative solution strategy via indirect approach:

- Household problem:

$$\max_{c,n} \; u(c,n)$$

subject to:

- Resource constraint

$$c + g = n \tag{4}$$

- Implementability constraint

$$u_c(c,n)c + u_n(c,n)n = 0 \tag{5}$$

A toy model to fix ideas of Ramsey optimality

Indirect approach

Comment: before solving this transformed problem, the equivalence of the two approaches needs to be checked!

Requirement: the allocations which satisfy the constraints (4) and (5) are identical to those satisfying the constraints (1), (2) and (3)

Let $A^{dir} = \{(c, n) : \exists\ \tau \in (0, 1)$ s.t. (1), (2), (3) are satisfied$\}$
and $A^{indir}\{(c, n)$ s.t. (4) and (5) are satisfied$\}$

$\longrightarrow$ Requirement for the indirect approach to be equivalent to the direct approach:

$$A^{dir} = A^{indir}$$

A toy model to fix ideas of Ramsey optimality

Indirect approach

$A^{dir} = A^{indir}$: Two steps to verify this statement

Step 1: $(c, n) \in A^{dir} \Rightarrow (c, n) \in A^{indir}$
Suppose $(c, n) \in A^{dir}$, ie $\exists\ \tau \in (0, 1)$ s.t. $c = (1 - \tau)n$,
$(1 - \tau)u_c(c, n) = -u_n(c, n)$ and $g = \tau n$

- Start out from $(1 - \tau)u_c(c, n) = -u_n(c, n)$ and multiply by $\frac{c}{1-\tau} = n$, implying the implementability constraint (5) holds
- Next, adding $c = (1 - \tau)n$ (ie private bc) and $g = \tau n$ (ie gov't bc) yields the resource constraint (4).

Step 2: $(c, n) \in A^{indir} \Rightarrow (c, n) \in A^{dir}$
Suppose $(c, n) \in A^{indir}$, ie $u_c(c, n)c + u_n(c, n)n = 0$ and $c + g = n$.

- First, fix τ such that $(1 - \tau) = \frac{-u_n(c,n)}{u_c(c,n)}$, implying HH optimality in (2).
- Multiply $(1 - \tau) = \frac{-u_n(c,n)}{u_c(c,n)}$ by n and use the implementability constraint (5) to establish $c = (1 - \tau)n$, ie the private bc (1)
- Finally, combine $c = (1 - \tau)n$ and $c + g = n$ to establish the gov't bc, ie eqn (3).

A toy model to fix ideas of Ramsey optimality

Indirect approach: parametric example

Parametric example:

$$\max_{c,n} \; u(c,n) = c - \frac{1}{2}n^2$$

subject to:

- Resource constraint

$$c + g = n$$

- Implementability constraint

where $u_c(c,n)c + u_n(c,n)n = 0$ turns into

$$c = n^2$$

A toy model to fix ideas of Ramsey optimality

Indirect approach: parametric example

Hence:

$$\max_{n} \ u(n) = \frac{1}{2}n^2 \ \ s.t. \ \ n^2 + g = n$$

- Constraint is quadratic in n, for given value of g, and solved by two values n_1 and n_2, with: $0 < n_1 < n_2$
- Objective $u(n) = \frac{1}{2}n^2$ rises monotonically in n, for $n > 0$

$\Rightarrow$

$$n^* = n_2 = \frac{1}{2} + (\frac{1}{4} - g)^{\frac{1}{2}} = \frac{1}{2} + \frac{1}{2}(1-4g)^{\frac{1}{2}}$$

which is the same result as under the direct approach, where we got:

$$\tau^* = \frac{1}{2} - \frac{1}{2}(1-4g)^{\frac{1}{2}}, \ \ n^*(\tau^*) = 1 - \tau^*, \ \ \ c^*(\tau^*) = (1-\tau^*)^2$$

MIU model: a static version

→ Let us use these concepts to address the debate between Friedman and Phelps

→ Starting point: static version of the MIU model
(see: Mulligan/Sala-i-Martin, 1997 and comment from Fisher 1997; we use the notation in line with Walsh, p. 176 ff.)

→ As it happens and notwithstanding the equivalence of the two approaches:

Direct approach: helps to see what drives the **Phelps-conjecture**
Indirect approach: helps to see what drives the **Friedman-conjecture**

MIU model: a static version

Model has 3 ingredients:

- Utility of representative agent

$$\max_{c,m,l} \; u(c, m, l) \tag{6}$$

- Private sector budget constraint:

$$f(1-l) = (1+\tau)c + \tau_m m \tag{7}$$

- Government budget constraint

$$g = \tau_m m + \tau c \tag{8}$$

Notation:

- l = leisure (and $1 - l$ = labour)
- 2 taxes: $\tau_m = i/(1+i)$ = tax on real balances, τ = consumption tax
- g = gov't spending, assumed constant

MIU model: a static version

Assumptions:

A1) Satiation
For any pair $c > 0$ and $l > 0$ there exists a finite satiation level of $\overline{m}$ such that $m|_{\tau_{m=0}} = \overline{m}$

A2) Define real revenues R^{gov} s.t.

$$R^{gov} \equiv \tau_m m + \tau c$$

and assume that both taxes τ_m and τ are on the upward sloping part of the Laffer curve: $\frac{\partial R^{gov}}{\partial \tau} > 0$, $\frac{\partial R^{gov}}{\partial \tau_m} > 0$

Restrictions on optimal policy:
$\tau > 0$, $\tau_m \geqslant 0$ (Friedman rule: $\tau_m = 0$)

MIU model: a static version

Characterization of optimal policy via direct approach:

Step 1: solve **HH optimality problem**:

$$\max_{c,m,l} \; u(c,m,l) + \lambda[f(1-l) - (1+\tau)c - \tau_m m] \tag{9}$$

to obtain demand functions $c(\tau_m,\tau)$, $m(\tau_m,\tau)$, $l(\tau_m,\tau)$ and use them to get the indirect utility function:

$$v(\tau_m,\tau) \equiv u(c(\tau_m,\tau), m(\tau_m,\tau), l(\tau_m,\tau))$$

Step 2: solve **Ramsey problem (government optimality problem)**:

$$\max_{\tau_m,\tau} \; v(\tau_m,\tau) + \mu[R^{gov}(\tau_m,\tau) - g] \tag{10}$$

with:

$$R^{gov}(\tau_m,\tau) = \tau_m m(\tau_m,\tau) + \tau c(\tau_m,\tau)$$

MIU model: a static version

Interior optimum $(\tau_m > 0)$ satisfies:

$$\frac{\frac{\partial v(\tau_m,\tau)}{\partial_{\tau_m}}}{\frac{\partial R^{gov}(\tau_m,\tau)}{\partial_{\tau_m}}} = \frac{\frac{\partial v(\tau_m,\tau)}{\partial_{\tau}}}{\frac{\partial R^{gov}(\tau_m,\tau)}{\partial_{\tau}}} \tag{11}$$

Interpretation:

- Arbitrage condition between the two instruments in terms of welfare
- LHS of (11) measures the marginal effect of the inflation tax on utility per euro of revenue raised.
  This should be equal to the RHS of (11) which measures the marginal effect of the consumption tax on utility per euro of revenue raised.

MIU model: a static version

Friedman rule ($\tau_m = 0$) captured by a corner solution
$\longrightarrow$ necessary condition for $\tau_m = 0$ to be optimal:

$$\left.\frac{\frac{\partial v(\tau_m,\tau)}{\partial_{\tau_m}}}{\frac{\partial R^{gov}(\tau_m,\tau)}{\partial_{\tau_m}}}\right|_{\tau_{m=0}} \leq \left.\frac{\frac{\partial v(\tau_m,\tau)}{\partial_{\tau}}}{\frac{\partial R^{gov}(\tau_m,\tau)}{\partial_{\tau}}}\right|_{\tau_{m=0}} \tag{12}$$

Interpretation:

- Notice: $\frac{\partial v(\tau_m,\tau)}{\partial_{\tau_m}} < 0$, $\frac{\partial v(\tau_m,\tau)}{\partial_{\tau}} < 0$
- Friedman rule can only be optimal if, at $\tau_m = 0$, the marginal welfare loss (in absolute terms) induced by the inflation tax per euro of revenue raised is larger than (or at least equal to) the marginal welfare loss induced by the consumption tax per euro of revenue raised.

MIU model: a static version

Phelps-conjecture

Phelps-conjecture ($\tau_m = 0$ is not optimal)

- For various preferences policy will be characterized by the interior optimum (11)
- This implies that the optimal tax system should tax both m and c at positive tax rates ($\tau_m > 0$, $\tau > 0$)
- In such system taxes should be set according to the **inverse elasticity rule**, ie: the consumption tax should be large relative to the inflation tax when $\varepsilon_{c,1+\tau}$ is small relative to ε_{m,τ_m}, and vice versa, where:
  $\varepsilon_{m,\tau_m} = \left| m_{\tau_m} \frac{\tau_m}{m} \right|$: interest elasticity of money demand
  $\varepsilon_{c,1+\tau} = \left| c_{1+\tau} \frac{1+\tau}{c} \right|$: price elasticity of consumption demand

MIU model: a static version

Phelps-conjecture

Phelps-conjecture: fragility

- Derivation of Phelps-conjecture is correct
- Yet, for a particular and widely used class of preferences optimal policy will be characterized by the corner solution (12)
- To get some indication why the corner solution can be attractive, consider the utility function used in previous Lectures, ie

$$u(c,m,l) = \widetilde{u}(c,m) + v(l) = \frac{\left[ac^{1-b} + (1-a)m^{1-b}\right]^{\frac{1-\Phi}{1-b}}}{1-\Phi} + \Psi \frac{l^{1-\eta}}{1-\eta} \tag{13}$$

- Recall that (13) does not satisfy A1 (Satiation) for a finite level of m, indicating that the analysis of $\tau_m = 0$ creates a special challenge
- Similarly, (13) may not always satisfy A2 ($\frac{\partial R^{gov}}{\partial \tau_m} > 0$). ($\longrightarrow$ For example consider the special case where $\widetilde{u}(c,m)$ is of Cobb-Douglas-type and leisure is entirely inelastic)

MIU model: a static version

Friedman-conjecture

Chari, Christiano and Kehoe (1996):

- If preferences are of type (13): Friedman rule is approximately valid
- Why only approximately? since $m|_{\tau_m=0} \longrightarrow \infty$, a solution will not exist for $\tau_m = 0$, but it will be optimal to make τ_m arbitrarily small
- What makes preferences of type (13) special? They satisfy two properties which are used by CCK (1996) to confirm the approximate validity of the Friedman rule:
  1) $u(c, m, l)$ is additively **separable** in i) c, m and ii) l
  2) $\widetilde{u}(c, m)$ is **homothetic** in c and m

MIU model: a static version

Friedman-conjecture

- Consider

$$u(c,m,l)=\frac{\left[ac^{1-b}+(1-a)m^{1-b}\right]^{\frac{1-\Phi}{1-b}}}{1-\Phi}+\Psi\frac{l^{1-\eta}}{1-\eta}$$

and assume

$$m \leq \widehat{m} \quad \text{(exogenous upper limit on } m \text{ via technology, with } i>0\text{)}$$

- Two ways to show (approximate) validity of Friedman rule
I) Direct approach: in general rather tedious
II) Indirect approach: rather straightforward and used by CCK (1996)

- Convenient simplification: **linear technology,** ie

$$f(1-l)\equiv 1-l \quad \text{such that} \quad f'(1-l)=w=1$$

MIU model: a static version

Friedman-conjecture

Indirect approach (general structure)

Go back to (6)-(8) and see that the indirect approach implies:

$$\max_{c,m,l} \; u(c, m, l) \quad \text{subject to:} \tag{14}$$

- Technology

$$m \leq \widehat{m} \tag{15}$$

- Resource constraint

$$c + g = 1 - l \tag{16}$$

  [which follows from combining the private sector bc:
  $1 - l = (1 + \tau)c + \tau_m m$
  and the gov't bc: $g = \tau_m m + \tau c$]

- Implementability constraint

$$u_c(c, m, l)c + u_m(c, m, l)m - u_l(c, m, l)(1 - l) = 0 \tag{17}$$

MIU model: a static version

Friedman-conjecture

Indirect approach (general structure)

Remark: to verify the implementability constraint (17), solve the HH optimality problem

$$\max_{c,m,l} \; u(c,m,l) \;\; s.t. \quad (1+\tau)c + \tau_m m = 1 - l$$

with FOC's conditions

$$\begin{aligned} \frac{u_c(c,m,l)}{u_l(c,m,l)} &= 1+\tau \\ \frac{u_m(c,m,l)}{u_l(c,m,l)} &= \tau_m \end{aligned}$$

and substitute these conditions back into the HH bc to obtain (17), ie:

$$u_c(c,m,l)c + u_m(c,m,l) - u_l(c,m,l)(1-l) = 0$$

MIU model: a static version

Friedman-conjecture

Indirect approach (special features of $u(c, m, l)$):

Consider

$$\widetilde{u}(c, m) = \frac{\left[ac^{1-b} + (1-a)m^{1-b}\right]^{\frac{1-\Phi}{1-b}}}{1-\Phi}$$

Rewrite it as

$$\widetilde{u}(c, m) = h(\widetilde{\widetilde{u}}(c, m))$$

with

$$\begin{aligned} h(x) &= \frac{x^{1-\Phi}}{1-\Phi}, \\ \widetilde{\widetilde{u}}(c, m) &= \left[ac^{1-b} + (1-a)m^{1-b}\right]^{\frac{1}{1-b}} \end{aligned}$$

Homotheticity:

i) The function $\widetilde{\widetilde{u}}(c, m)$ is homogenous of degree 1 in c and m, because of $\widetilde{\widetilde{u}}(\alpha c, \alpha m) = \alpha \widetilde{\widetilde{u}}(c, m))$

ii) The function $h(.)$ induces a monotone transformation of $\widetilde{\widetilde{u}}(c, m)$

MIU model: a static version

Friedman-conjecture

Indirect approach (solution steps):

Let us solve (14)-(17), but assume, for the time being, $m \leq \widehat{m}$ is slack:

$$\begin{aligned}
&\max_{c,m,l} \ \widetilde{u}(c,m) + v(l) \\
&+\gamma(1-l-c-g) \\
&+\lambda[\widetilde{u}_c(c,m)\cdot c + \widetilde{u}_m(c,m)\cdot m - v_l(l)\cdot(1-l)]
\end{aligned}$$

FOC's w.r.t. c, m, l

$$\begin{aligned}
(1+\lambda)\widetilde{u}_c + \lambda[\widetilde{u}_{cc}c + \widetilde{u}_{mc}m] &= \gamma \\
(1+\lambda)\widetilde{u}_m + \lambda[\widetilde{u}_{cm}c + \widetilde{u}_{mm}m] &= 0 \\
(1+\lambda)v_l - \lambda v_{ll}(1-l) &= \gamma
\end{aligned}$$

MIU model: a static version
Friedman-conjecture

Indirect approach (solution steps):

Rewrite the FOC's w.r.t. c and m

$$\widetilde{u}_c[1+\lambda+\lambda\frac{\widetilde{u}_{cc}c+\widetilde{u}_{mc}m}{\widetilde{u}_c}] = \gamma \qquad (18)$$

$$\widetilde{u}_m[1+\lambda+\lambda\frac{\widetilde{u}_{cm}c+\widetilde{u}_{mm}m}{\widetilde{u}_m}] = 0. \qquad (19)$$

Hint (see Appendix): Homotheticity implies:

$$\frac{\widetilde{u}_{cc}c+\widetilde{u}_{mc}m}{\widetilde{u}_c} = \frac{\widetilde{u}_{cm}c+\widetilde{u}_{mm}m}{\widetilde{u}_m}.$$

$\longrightarrow$ Division of (19) by (18) yields:

$$\frac{\widetilde{u}_m}{\widetilde{u}_c} = 0$$

MIU model: a static version

Friedman-conjecture

Indirect approach (interpretation):

- Remember that the optimality condition

$$\frac{\tilde{u}_m}{\tilde{u}_c} = 0 \qquad (20)$$

  has been derived for the unconstrained problem, assuming $m < \widehat{m}$.

- How to choose τ_m and τ such that the optimality condition can be implemented? As derived above, private sector optimal behaviour implies

$$\frac{\tilde{u}_m}{\tilde{u}_c} = \frac{\tau_m}{1+\tau}$$

$\rightarrow$ **(Approximate) Validity of Friedman rule:**
In view of (20), for any finite value $\widehat{m}$ (with associated $i > 0$) optimality requires $m = \widehat{m}$, but as the bound $\widehat{m}$ increases, (20) implies that the optimal interest rate converges against zero: $\tau_m = \frac{i}{1+i} \rightarrow 0$.

Cash-in- advance-model: intertemporal analysis

Idea: extend the debate on Phelps vs. Friedman in two dimensions:

1) MIU model is evidently a short-cut with unclear/missing micro-foundations.
$\longrightarrow$ Let us consider a **cash-in-advance economy**, ie a widely used alternative.
2) The analysis so far has been static.
Let us consider a fully developed **intertemporal** CIA model.

$\longrightarrow$ to be shown, in line with CCK (1996):
For some CIA-economies, **Friedman rule** can be supported (using the indirect approach), under assumptions with respect to preferences similar to the just discussed ones: **homotheticity** and **separability**

Cash-in- advance-model: intertemporal analysis

General remarks

Motivation behind CIA-model is simple:

- In advanced economies, most transactions do not require direct money holdings ('cash'); but for some transactions cash is still essential
- CIA model captures coexistence of cash and non-cash payments by two types of goods, so-called cash goods and credit goods
- Nominal interest rate acts like a tax on cash goods, while credit goods are immune against this (ie they are paid by drawing on interest-bearing assets)

Intuition for main finding: Friedman rule helps to ensure uniform taxation between cash and credit goods

Cash-in- advance-model: intertemporal analysis

General remarks

(Simplifying) **features** of the particular CIA-economy to be analyzed:

- deterministic set up
- taxation of labour (τ^h) vs. taxation of real balances
- timing of transactions and payments in line with Svensson (1985): $\longrightarrow$ within the representative period: good markets open first, prior to asset markets
- Let: c_{1t} = cash good, c_{2t} = credit good
- Homotheticity in c_{1t}, c_{2t} and additive separability w.r.t. l_t:

$$u(c_{1t}, c_{2t}, l_t) = \tilde{u}(c_{1t}, c_{2t}) + v(l_t)$$

$\longrightarrow$ notation: see Walsh p. 99f. and 182 f., but let $\tau^c = 0$, $Q_t = P_t$

Cash-in- advance-model: intertemporal analysis

Model ingredients

Objective of representative household:

$$\max_{c_{1t}, c_{2t}, l_t} \sum_{t=0}^{\infty} \beta^t [\widetilde{u}(c_{1t}, c_{2t}) + v(l_t)] \tag{21}$$

Household budget constraint:

$$P_t(c_{1t} + c_{2t}) + M_t + B_t = \left(1 - \tau_t^h\right) P_t(1 - l_t) + (1 + i_{t-1})B_{t-1} + M_{t-1} \tag{22}$$

CIA constraint:

$$P_t c_{1t} \leq M_{t-1} \tag{23}$$

Government budget constraint:

$$M_t + B_t = (1 + i_{t-1})B_{t-1} + M_{t-1} + P_t g_t - \tau_t^h P_t(1 - l_t) \tag{24}$$

Cash-in- advance-model: intertemporal analysis

Model ingredients: comments

1) Cash-in-advance-constraint:

- for both B and M being voluntarily held in equilibrium: $i \geqslant 0$
- if $i > 0$: CIA-constraint will be strictly binding $Pc_1 = M$
- if $i = 0$: we make the (innocuous) tiebreaker assumption that $Pc_1 = M$ continues to hold

2) Initial values:

- Economy starts in $t = 0$, taking as given B_{-1}, M_{-1}, i_{-1}
- Assumption: $A_{-1} \equiv B_{-1} + M_{-1} = 0$

Cash-in- advance-model: intertemporal analysis

Model ingredients:

→ Express the constraints (22)-(24) in real terms (deflated by P_t)

Household budget constraint:

$$(c_{1t} + c_{2t}) + m_t + b_t = \left(1 - \tau_t^h\right)(1 - l_t) + (1 + r_{t-1})b_{t-1} + \frac{m_{t-1}}{1 + \pi_t} \tag{25}$$

CIA-constraint:

$$c_{1t} = \frac{m_{t-1}}{1 + \pi_t} \tag{26}$$

Government budget constraint:

$$m_t + b_t = (1 + r_{t-1})b_{t-1} + \frac{m_{t-1}}{1 + \pi_t} + g_t - \tau_t^h(1 - l_t) \tag{27}$$

memo:

$$\begin{aligned} (1 + i_{t-1})\frac{B_{t-1}}{P_t} &= (1 + i_{t-1})\frac{B_{t-1}}{P_{t-1}}\frac{P_{t-1}}{P_t} = (1 + r_{t-1})b_{t-1} \\ \frac{M_{t-1}}{P_t} &= \frac{M_{t-1}}{P_{t-1}}\frac{P_{t-1}}{P_t} = \frac{m_{t-1}}{1 + \pi_t} \end{aligned}$$

Cash-in- advance-model: intertemporal analysis

Optimal policy via indirect approach: overview

Optimality problem to be solved under the indirect approach:

$$\max_{c_{1t},c_{2t},l_t} \sum_{t=0}^{\infty} \beta^t [\widetilde{u}(c_{1t}, c_{2t}) + v(l_t)] \tag{28}$$

- Resource constraint

$$1 - l_t = c_{1t} + c_{2t} + g_t \tag{29}$$

- Implementability constraint

$$\sum_{t=0}^{\infty} \beta^t [\widetilde{u}_{c_{1t}} c_{1t} + \widetilde{u}_{c_{2t}} c_{2t} - v_{l_t}(1 - l_t)] = 0 \tag{30}$$

- Portfolio constraint (from $i \geqslant 0$)

$$\frac{\widetilde{u}_{c_{1t}}}{\widetilde{u}_{c_{2t}}} \geq 1 \tag{31}$$

Cash-in- advance-model: intertemporal analysis

Optimal policy via indirect approach: verifying the constraints

$\longrightarrow$ Before we solve this problem, let us confirm why it is subject to the 3 constraints (29)-(31)
$\longrightarrow$ **But:** the logic behind this is the same as for the static MIU model!

I) Confirmation of resource constraint (29):
For all $t \geqslant 0$, the resource constraint

$$1 - l_t = c_{1t} + c_{2t} + g_t$$

follows from combining the flow HH budget constraint (25)

$$(c_{1t} + c_{2t}) + m_t + b_t = \left(1 - \tau_t^h\right)(1 - l_t) + (1 + r_{t-1})b_{t-1} + \frac{m_{t-1}}{1 + \pi_t}$$

and the flow gov 't budget constraint (27)

$$m_t + b_t = (1 + r_{t-1})b_{t-1} + \frac{m_{t-1}}{1 + \pi_t} + g_t - \tau_t^h(1 - l_t)$$

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II) Confirmation of implementability constraint (30):
$\longrightarrow$ Idea: use the FOC's from HH optimality problem to substitute out for all terms containing prices and taxes in the HH bc

$\longrightarrow$ need to solve: HH optimality problem

$$\max_{c_{1t}, c_{2t}, l_t, m_t, b_t,} : \quad \sum_{t=0}^{\infty} \beta^t [\widetilde{u}(c_{1t}, c_{2t}) + v(l_t)$$

$$+\psi_t \left(\left(1 - \tau_t^h\right)(1 - l_t) + (1 + r_{t-1})b_{t-1} + \frac{m_{t-1}}{1+\pi_t} - (c_{1t} + c_{2t}) - m_t - b_t \right)$$

$$+\mu_t \left(\frac{m_{t-1}}{1+\pi_t} - c_{1t} \right)]$$

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From the FOC's

$$\begin{aligned}
\widetilde{u}_{c_{1t}} &= \psi_t + \mu_t \\
\widetilde{u}_{c_{2t}} &= \psi_t \\
v_{l_t} &= \psi_t (1 - \tau_t^h) \\
\psi_t &= \beta \psi_{t+1} (1 + r_t) \\
\psi_t &= \beta \frac{\psi_{t+1} + \mu_{t+1}}{1 + \pi_{t+1}}
\end{aligned}$$

we get

$$\frac{v_{l_t}}{\widetilde{u}_{c_{2t}}} = 1 - \tau_t^h$$

$$\frac{\widetilde{u}_{c_{1t}}}{\widetilde{u}_{c_{2t}}} = \frac{\psi_t + \mu_t}{\psi_t} = (1 + r_{t-1})(1 + \pi_t) = 1 + i_{t-1} \tag{32}$$

$$\frac{1}{\beta} \frac{\widetilde{u}_{c_{2t-1}}}{\widetilde{u}_{c_{2t}}} = 1 + r_{t-1}$$

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Why did we establish

$$\textbf{I)}\ 1-\tau_t^h = \frac{v_{l_t}}{\widetilde{u}_{c_{2t}}} \qquad \textbf{II)}\ 1+i_{t-1} = \frac{\widetilde{u}_{c_{1t}}}{\widetilde{u}_{c_{2t}}} \qquad \textbf{III)}\ 1+r_{t-1} = \frac{1}{\beta}\frac{\widetilde{u}_{c_{2t-1}}}{\widetilde{u}_{c_{2t}}},$$

ie 3 expressions for $1-\tau_t^h$, $1+i_{t-1}$, and $1+r_{t-1}$?

Because the flow HH budget constraint

$$\left(1-\tau_t^h\right)(1-l_t)+(1+r_{t-1})b_{t-1}+\frac{m_{t-1}}{1+\pi_t} = c_{1t}+c_{2t}+m_t+b_t,$$

can be rearranged as (if one uses $a_t \equiv m_t + b_t$ and $\frac{m_{t-1}}{1+\pi_t} = c_{1t}$):

$$\left(1-\tau_t^h\right)(1-l_t)+(1+r_{t-1})a_{t-1}-\frac{i_{t-1}}{1+\pi_t}m_{t-1} = c_{1t}+c_{2t}+a_t$$

$$\underbrace{(1+r_{t-1})}_{\text{use III}}a_{t-1} = \underbrace{(1+i_{t-1})}_{\text{use II}}c_{1t}+c_{2t}-\underbrace{\left(1-\tau_t^h\right)}_{\text{use I}}(1-l_t)+a_t \qquad (33)$$

Cash-in- advance-model: intertemporal analysis

Optimal policy via indirect approach: verifying the constraints

→ Start out from (33), ie

$$\underbrace{(1+r_{t-1})}_{\text{use III}}a_{t-1} = \underbrace{(1+i_{t-1})}_{\text{use II}}c_{1t} + c_{2t} - \underbrace{\left(1-\tau_t^h\right)}_{\text{use I}}(1-l_t) + a_t$$

to get $$\frac{1}{\beta}\frac{\widetilde{u}_{c_{2t-1}}}{\widetilde{u}_{c_{2t}}}a_{t-1} = \frac{\widetilde{u}_{c_{1t}}}{\widetilde{u}_{c_{2t}}}c_{1t} + c_{2t} - \frac{v_{l_t}}{\widetilde{u}_{c_{2t}}}(1-l_t) + a_t$$

⇔

$$\widetilde{u}_{c_{2t-1}}a_{t-1} = \beta\left[\widetilde{u}_{c_{1t}}c_{1t} + \widetilde{u}_{c_{2t}}c_{2t} - v_{l_t}(1-l_t) + \widetilde{u}_{c_{2t}}a_t\right] \tag{34}$$

→ Note: $A_{-1} = B_{-1} + M_{-1} = 0$ and $\widetilde{u}_{c_{2,t-1}}\big|_{t=0}$ is not a choice variable (ie allocations of period -1 are given and cannot be chosen in $t = 0$)

→ Get a special condition for $t = 0$ and combine this with (34) for all $t \geqslant 1$ to obtain via forward iteration the implementability constraint (30), ie

$$\sum_{t=0}^{\infty}\beta^t\left[\widetilde{u}_{c_{1t}}c_{1t}+\widetilde{u}_{c_{2t}}c_{2t} - v_{l_t}(1-l_t)\right] = 0$$

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Optimal policy via indirect approach: verifying the constraints

$\rightarrow$ $t = 0$: $A_{-1} = B_{-1} + M_{-1} = 0$ implies

$$0 = \widetilde{u}_{c_{1,0}} c_{1,0} + \widetilde{u}_{c_{2,0}} c_{2,0} - v_{l_0}(1 - l_0) + \widetilde{u}_{c_{2,0}} a_0$$

$\rightarrow$ $t = 1$: to substitute out for $\widetilde{u}_{c_{2,0}} a_0$, use condition (34) at $t = 1$, ie

$$\widetilde{u}_{c_{2,0}} a_0 = \beta \left[\widetilde{u}_{c_{1,1}} c_{1,1} + \widetilde{u}_{c_{2,1}} c_{2,1} - v_{l_1}(1 - l_1) + \widetilde{u}_{c_{2,1}} a_1 \right]$$

$\rightarrow$ $t \geq 2$: make repeated substitutions to verify the implementability constraint (30), ie

$$\sum_{t=0}^{\infty} \beta^t [\widetilde{u}_{c_{1t}} c_{1t} + \widetilde{u}_{c_{2t}} c_{2t} - v_{l_t}(1 - l_t)] = 0$$

Cash-in- advance-model: intertemporal analysis

Optimal policy via indirect approach: verifying the constraints

III) Confirmation of portfolio constraint (31):

From the FOC's we had established (32), ie

$$\frac{\widetilde{u}_{c_{1t}}}{\widetilde{u}_{c_{2t}}} = 1 + i_{t-1}$$

Because of $i \geqslant 0$, we get (31), ie

$$\frac{\widetilde{u}_{c_{1t}}}{\widetilde{u}_{c_{2t}}} \geq 1$$

Cash-in- advance-model: intertemporal analysis

Characterization of optimal policy via indirect approach

Indirect approach: solution of (28)-(31)

→ 'Unconstrained problem': ignore, for the time being, $\frac{\widetilde{u}_{c_{1t}}}{\widetilde{u}_{c_{2t}}} \geq 1$

$$\max_{c_{1t}, c_{2t}, l_t} \sum_{t=0}^{\infty} \beta^t [\widetilde{u}(c_{1t}, c_{2t}) + v(l_t) \\ +\gamma_t(1 - l_t - c_{1t} - c_{2t} - g_t) \\ +\lambda_t\left(\widetilde{u}_{c_{1t}} c_{1t} + \widetilde{u}_{c_{2t}} c_{2t} - v_{l_t}(1 - l_t)\right)]$$

→ This problem has a quasi-static structure, ie the solution will be similar to the static MIU-model solved above

FOC's w.r.t. c_{1t}, and c_{2t} (with short-cut: $\widetilde{u}_{c_{1t}} \equiv \widetilde{u}_1, \widetilde{u}_{c_{2t}} = \widetilde{u}_2$)

$$\begin{aligned} \widetilde{u}_1[1 + \lambda_t + \lambda_t \frac{\widetilde{u}_{11} c_{1t} + \widetilde{u}_{21} c_{2t}}{\widetilde{u}_1}] &= \gamma_t \\ \widetilde{u}_2[1 + \lambda_t + \lambda_t \frac{\widetilde{u}_{12} c_{1t} + \widetilde{u}_{22} c_{2t}}{\widetilde{u}_2}] &= \gamma_t \end{aligned}$$

Cash-in- advance-model: intertemporal analysis

Characterization of optimal policy via indirect approach

Indirect approach: solution of (28)-(31)

The assumed homotheticity of $\widetilde{u}$ in c_{1t}, and c_{2t} implies ($\longrightarrow$ *see: Appendix*)

$$\frac{\widetilde{u}_{11} c_{1t} + \widetilde{u}_{21} c_{2t}}{\widetilde{u}_1} = \frac{\widetilde{u}_{12} c_{1t} + \widetilde{u}_{22} c_{2t}}{\widetilde{u}_2},$$

ensuring that optimal policy is characterized by

$$\frac{\widetilde{u}_{c_{1t}}}{\widetilde{u}_{c_{2t}}} = 1 \tag{35}$$

Main result: This optimality criterion:

$\longrightarrow$ can be achieved under the constrained problem (which respects $\frac{\widetilde{u}_{c_{1t}}}{\widetilde{u}_{c_{2t}}} \geq 1$)

$\longrightarrow$ and it can be implemented with $i = 0$ (ie the Friedman rule) in all periods

Appendix: Homothetic preferences

- Let us rewrite the HH optimality problem of the MIU model in standard notation from consumer theory

$$\max_{c,m,l} : \quad \widetilde{u}(c,m) + v(l) \quad \text{s.t.:} \quad \underbrace{p^c}_{1+\tau} \cdot c + \underbrace{p^m}_{\tau_m} \cdot m + \underbrace{w}_{1} \cdot l = \underbrace{w}_{1}$$

- For given prices p^c, p^m and a given wage rate w, this problem is solved by the (Marshallian) demand functions

$$c^* = c(p^c, p^m, w), \quad m^* = m(p^c, p^m, w), \quad l^* = l(p^c, p^m, w)$$

- **Definition:** Preferences are homothetic in c and m iff

$$\frac{\partial c^*}{\partial w}\frac{w}{c^*} = \frac{\partial m^*}{\partial w}\frac{w}{m^*}, \tag{36}$$

ie an increase in w leads to equiproportionate changes in c^* and m^* (such that the ratio between c^* and m^* remains constant along the income expansion path)

- **Claim:** Homotheticity in c and m implies

$$\frac{\widetilde{u}_{cc}c^* + \widetilde{u}_{mc}m^*}{\widetilde{u}_c} = \frac{\widetilde{u}_{cm}c^* + \widetilde{u}_{mm}m^*}{\widetilde{u}_m}. \tag{37}$$

Appendix: Homothetic preferences

- To verify (37) consider

$$\max_{c,m,l,\mu} : \ \widetilde{u}(c,m) + v(l) + \mu \cdot [\underbrace{p^c}_{1+\tau} \cdot c + \underbrace{p^m}_{\tau_m} \cdot m + \underbrace{w}_{1} \cdot l - \underbrace{w}_{1}]$$

  and add

$$\mu^* = c(p^c, p^m, w)$$

  to the (Marshallian) demand functions

- System of optimized first-order conditions:

$$\begin{aligned} p^c c^* + p^m m^* - w(1 - l^*) &\equiv 0 \\ \widetilde{u}_c(c^*, m^*) + \mu^* p^c &\equiv 0 \\ \widetilde{u}_m(c^*, m^*) + \mu^* p^m &\equiv 0 \\ v_l(l^*) + \mu^* w &\equiv 0 \end{aligned}$$

Appendix: Homothetic preferences

- To use (36) for the verification of (37) we need to get $\frac{\partial c^*}{\partial w}$ and $\frac{\partial m^*}{\partial w}$
- Differentiating of the system of optimized first-order conditions with respect to w yields:

$$\underbrace{\begin{bmatrix} 0 & p^c & p^m & w \\ p^c & \widetilde{u}_{cc} & \widetilde{u}_{cm} & 0 \\ p^m & \widetilde{u}_{mc} & \widetilde{u}_{mm} & 0 \\ w & 0 & 0 & v_{ll} \end{bmatrix}}_{H} \cdot \begin{bmatrix} \frac{\partial \mu^*}{\partial w} \\ \frac{\partial c^*}{\partial w} \\ \frac{\partial m^*}{\partial w} \\ \frac{\partial l^*}{\partial w} \end{bmatrix} = \begin{bmatrix} 1 - l^* \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{38}$$

Appendix: Homothetic preferences

- Applying Cramer's rule to (38) yields:

$$
\begin{aligned}
\frac{\partial c^*}{\partial w} &= -\frac{1-l^*}{|H|} \cdot \begin{vmatrix} p^c & \widetilde{u}_{cm} & 0 \\ p^m & \widetilde{u}_{mm} & 0 \\ w & 0 & v_{ll} \end{vmatrix} \\
&= -\frac{1-l^*}{|H|} v_{ll} \left(p^c \widetilde{u}_{mm} - p^m \widetilde{u}_{cm} \right) \\
&= \frac{1-l^*}{|H|} v_{ll} p^m \left(\widetilde{u}_{cm} - \frac{p^c}{p^m} \widetilde{u}_{mm} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial m^*}{\partial w} &= \frac{1-l^*}{|H|} \cdot \begin{vmatrix} p^c & \widetilde{u}_{cc} & 0 \\ p^m & \widetilde{u}_{mc} & 0 \\ w & 0 & v_{ll} \end{vmatrix} \\
&= \frac{1-l^*}{|H|} v_{ll} p^m \left(\frac{p^c}{p^m} \widetilde{u}_{mc} - \widetilde{u}_{cc} \right)
\end{aligned}
$$

Appendix: Homothetic preferences

- Hence, (36), ie

$$\frac{\partial c^*}{\partial w}\frac{w}{c^*} = \frac{\partial m^*}{\partial w}\frac{w}{m^*},$$

implies (where we use $\frac{p^c}{p^m} = \frac{\widetilde{u}_c}{\widetilde{u}_m}$)

$$\frac{\widetilde{u}_{cm} - \frac{\widetilde{u}_c}{\widetilde{u}_m}\widetilde{u}_{mm}}{c^*} = \frac{\frac{\widetilde{u}_c}{\widetilde{u}_m}\widetilde{u}_{mc} - \widetilde{u}_{cc}}{m^*}$$

which is equivalent to (37), ie

$$\frac{\widetilde{u}_{cc}\,c^* + \widetilde{u}_{mc}\,m^*}{\widetilde{u}_c} = \frac{\widetilde{u}_{cm}\,c^* + \widetilde{u}_{mm}\,m^*}{\widetilde{u}_m}.$$