Exam

Applied Intertemporal Optimization

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Frankfurt, February 18th 2010

1 A principal-agent model

There are two individuals, the principal and the agent. The agent lives T periods and at each period $t \in \{1, \ldots, T\}$ he works $n_t \geq 0$ and consumes $c_t \geq 0$. For each unit of labor he obtains one unit of income that can be spent in any period. The agent is myopic and has an instantaneous utility function v_t given by

$$v_t = \theta_t u(c_t) - n_t$$

with $u' > 0$ and $u'' < 0$, where $\theta_t \in [\underline{\theta}, \overline{\theta}]$ is privately known.

The principal is forward-looking but not fully informed as he does not know the true value of θ_t. He maximizes expected intertemporal utility of the agent

$$S = E_1 \Sigma_{t=1}^{T} v_t$$

given a dynamic budget constraint for wealth a_t,

$$a_{t+1} = (1 + r_t)(a_t + n_t - c_t),$$

with a positive interest rate r_t.

You are the principal and you would like to solve the maximisation problem by using dynamic programming. The control variable is c_t only.

1. Formulate the Bellman equation and derive an Euler equation for this problem.

2. Provide an interpretation to this equation.

2 A two period model

Considering a similar setup to above, let the individuals now live for two periods only. In each period $t \in \{1, 2\}$ the agent works $n_t \in [0, \overline{n}]$ and consumes $c_t \geq 0$. Instantaneous utility is again $v_t = \theta_t u(c_t) - n_t$. For each unit of labor he obtains one unit of income. Income can be stored and a positive interest rate r is paid on savings. The intertemporal budget constraint is

$$n_1(1 + r) + n_2 = c_1(1 + r) + c_2.$$

Consider first the case of complete information. Assuming the absence of discounting, the principal's utility function then reads

$$U = v_1 + v_2$$

with $u' > 0$ and $u'' < 0$, where $\theta_t \in [\underline{\theta}, \overline{\theta}]$ is known.

Solve the maximisation problem of the Principal using the Lagrangian.

1. Compute the first-order conditions, when control variables are both c_t and n_t.
2. Do you obtain an interior solution for n_1?
3. Deduce the relation between marginal utility from consumption today and tomorrow. Interpret your result. Can you derive explicit expressions for c_t, if utility is given by $u(c_t) = \ln c_t$?

   Now consider the case of incomplete information.

4. How does the utility function of the principal look like?
5. Compute optimal consumption and labour supply levels from the perspective of the principal, where utility is given again by $u(c_t) = \ln c_t$.
6. Does utility under incomplete information rise?

3 Simple RA-model

In a rational addiction (RA) model, the consumer's problem is to maximize

$$\int_0^T U(C(\tau), A(\tau), D(\tau))e^{-r\tau} d\tau, \tag{1}$$

subject to the instantaneous budget constraint

$$Y = C(\tau) + pA(\tau),$$

where $A(\tau)$ is consumption of the addictive good at time τ, $C(\tau)$ is consumption of non-addictive goods at τ and $D(\tau)$ is the stock of addiction of past consumption of A. The derivatives satisfy $U_A, U_C > 0$ and $U_D < 0$. The consumer maximizes (1) subject to the equation of motion for D:

$$\dot{D} = A - \delta D,$$

where δ, $0 < \delta < 1$, is the instantaneous rate of decay of the stock of addiction, which will be treated as a constant, which is independent of time.

1. Solve the problem above, using the Hamiltonian.
2. Derive a differential equation for A using the derived optimality conditions.
3. Draw a phase diagram with A on the y-axis and D on the x-axis, where the differential equations are given by

   $$\dot{D} = A - \delta D \quad \text{and} \quad \dot{A} = \frac{U_D - [r + \delta][pU_C - U_A]}{p^2 U_{CC} + U_{AA}},$$

   whereas the utility function U is given by

   $$U = \ln C + A - D^2.$$

4. Provide an economic interpretation of the phase diagram. Think of A as smoking and D as coughing from lung tar.