Lecture 6
Economic Growth: Optimal growth model and endogenous growth

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I What is missing? Topics in modern growth theory

Goal:

- This lecture gives an introduction to various questions addressed in modern growth theory
- As shown in the previous lecture from a number of perspectives, the ability of the Solow-model, when narrowly specified, to account quantitatively for empirical growth patterns both within and between countries is not fully satisfactory

Consequently, **new lines of research** have emerged which:

$\rightarrow$ **1)** modify the role of capital in the original Solow model

$\rightarrow$ **2)** focus on different and endogenously derived sources of growth

$\rightarrow$ **3)** allow for a more nuanced discussion of convergence

I What is missing? Topics in modern growth theory

1) Modifying the role of capital:

Starting point: Quantitative implications of Solow-model are not satisfactory if one assumes that the contribution of capital to output is fully captured by the private return earned by physical capital

(Main) **Extensions:**
(i) Accumulation of physical capital subject to **externalities** such that the economy-wide (or social) return to capital exceeds the private return
(ii) Parallel focus on **human and physical capital** allows for a more encompassing role of capital

→ Such extensions do a much better job to account for cross-country variations in per capita incomes

I What is missing? Topics in modern growth theory

2) Focus on different and endogenously derived sources of growth

Starting point:
→ Solow-model treats the 'effectiveness of labour' (ie A_t) as a black box: the core variable which explains long-run per capita growth is exogenously given
→ This approach is not satisfactory: growth should not occur by assumption, but it should rather be endogenously explained within the model - in particular, in view of lasting growth differentials between countries which seek explanation

Model extensions allow for various interpretations of A_t relating, among other things, to the diffusion of knowledge, the role of education and skills, the role of property rights, the quality of infrastructure etc.

→ In line with such interpretations, **endogenous growth theory** establishes mechanisms which replace the exogeneity of A_t against alternative engines of long-run growth, linked to country-specific fundamentals and variables that can be affected by policy decisions

I What is missing? Topics in modern growth theory

2) Focus on different and endogenously derived sources of growth

Comment:

- As recognized by the first generation of endogenous growth models, extensions of type 1) and 2) can well be combined, ie the accumulation of capital, when sufficiently broadly modelled, can maintain long-run per capita growth in the absence of growth through A_t
- Key requirement: the broad measure of capital must not be subject to diminishing (social) returns, but rather to constant returns to scale (*see:* Jones and Manuelli, 1990; Lucas, 1988; Rebelo, 1991; Romer, 1986)

I What is missing? Topics in modern growth theory

3) More nuanced discussion of convergence

Starting point: The absence of absolute convergence of per capita incomes is consistent with two alternative views:

View I**): Absence of any convergence** or, alternatively,
View II**): Existence of conditional convergence**

View I**): Absence of any convergence:**
$\rightarrow$ Countries can grow permanently at different per capita growth rates

I What is missing? Topics in modern growth theory

3) More nuanced discussion of convergence

View II) : Existence of conditional convergence
$\longrightarrow$ Countries are characterized by country-specific fundamentals (like savings rates and income shares) and, hence, by country-specific balanced growth paths
$\longrightarrow$ Convergence takes place conditional on these country-specific fundamentals

Example 1: Consider 2 'poor countries' with identical starting positions in terms of capital endowments per worker (ie K/N), but different savings rates.
$\longrightarrow$ The countries will have different growth rates during the catching-up phase
$\longrightarrow$ Conditional on the different values of s (which imply different steady-state values $k^{\#}_{So}(s_i)$, $i = 1, 2$), the statement that 'poor countries should grow faster than rich countries' remains correct

I What is missing? Topics in modern growth theory

3) More nuanced discussion of convergence

View II) : Existence of conditional convergence

Example 2: The hypothesis of absolute convergence has some support for a small number of OECD-countries for the post WW-II catching-up phase (but not for larger samples of countries)
$\rightarrow$ Since these countries are fundamentally very similar, this observation is consistent with conditional convergence

Implication of view II: in the long run, countries display identical per capita growth rates, but different levels of per capita incomes

I What is missing? Topics in modern growth theory

3) More nuanced discussion of convergence

View I vs. View II:
To distinguish between these 2 views empirically is challenging since fundamentals themselves may change over time

'Controversy':
$\longrightarrow$ In those poor countries which fail to catch-up fundamentals may have worsened, while they may have further improved in a number of countries which are already rich
notice: this interpretation (which emphasizes the role of transitional dynamics) is in the spirit of the Solow-model
$\longrightarrow$ At the same time: rich countries may benefit from permanently higher per capita growth rates
notice: this interpretation is in the spirit of endogenous growth theories

I What is missing? Topics in modern growth theory

Comment:

$\longrightarrow$ we will return to the concept of endogenous growth at the end of this lecture

$\longrightarrow$ but we will analyze first the ‘standard’ model of optimal growth with labour-augmenting technological progress and compare it with the Solow-model

$\longrightarrow$ this is advisable, since most models of endogenous growth are variations of the ‘standard’ optimal growth model

II Optimal growth model

- Let us reconsider the **standard neoclassical model of optimal growth**, now allowing for **exogenous population growth** ($\mu_N > 0$) and **labour-augmenting technological progress** ($\mu_A > 0$). The analysis goes back to Ramsey (1928), Cass (1965) and Koopmans (1965).
- **Objective**:

$$V_0 = \sum_{t=0}^{\infty} \beta^t U(c_t) \cdot N_t \tag{1}$$

- **Dynamic constraint** (aggregate economy):

$$K_{t+1} = F(K_t, N_t^{\#}) - C_t + (1-\delta)K_t, \tag{2}$$

with the predetermined value K_0 to be taken as given

II Optimal growth model

Comment on the objective, ie eqn (1):

$$V_0 = \sum_{t=0}^{\infty} \beta^t U(c_t) \cdot N_t$$

- The objective is derived from an individual welfare measure (ie the flow utility function is based on per capita consumption $c_t = \frac{C_t}{N_t}$)
- We allow for discounting via $\beta \in (0,1)$ to capture the impatience of the representative household. By allowing for the factor N_t, the representative household is assumed to grow over time at the rate $\mu_N > 0$. Hence, the objective mimics the behaviour of a 'dynasty' with perfect altruism between generations
- *Alternative interpretation:* the objective stands for a **Benthamite welfare function**, ie in **utilitarian spirit** the social planner takes into account that the population grows over time. This is equivalent to reducing the pure discount effect as captured by β, since it increases the weight given to the utility of a representative individual in the (distant) future
- *Notice:* This is different from **Ramsey** (1928) who argued that, as concerns the structure of the social planner's problem, there is no ethical case for discounting the future.

II Optimal growth model

Choice of the utility function:

- For better tractability let us use the particular utility function

$$U(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} \quad \text{with: } \sigma > 0 \text{ and } \sigma \neq 1$$

  where $\frac{1}{\sigma}$ denotes the intertemporal elasticity of substitution in consumption

- To ensure that the objective (1) has a well-defined solution (in the sense that lifetime utilities do not diverge) we impose the parameter restriction

$$\theta - \mu_N - (1-\sigma)\mu_A > 0, \tag{3}$$

  using $\beta = (1+\theta)^{-1}$. The role of this restriction will become clear below

II Optimal growth model

Transformation of c into units of effective labour:

- From the analysis of the Solow-model we anticipate that the steady-state solution will involve variables expressed in terms of units of effective labour
- Hence, we use

$$c_t = A_t \cdot c_t^{\#} = (1 + \mu_A)^t \cdot A_0 \cdot c_t^{\#}$$

II Optimal growth model

- Incorporating these features, the **objective** (1) can be rewritten as

$$\begin{aligned} & \sum_{t=0}^{\infty} \beta^t \cdot \frac{(c_t)^{1-\sigma} - 1}{1-\sigma} \cdot N_t \\ = & \sum_{t=0}^{\infty} \beta^t \cdot \frac{((1+\mu_A)^t \cdot A_0 \cdot c_t^{\#})^{1-\sigma} - 1}{1-\sigma} \cdot N_t \\ = & \sum_{t=0}^{\infty} [\beta \cdot (1+\mu_N)]^t \cdot \frac{((1+\mu_A)^t \cdot A_0 \cdot c_t^{\#})^{1-\sigma} - 1}{1-\sigma} \cdot N_0, \qquad (4) \end{aligned}$$

  where the last step uses

$$N_t = (1+\mu_N)^t \cdot N_0$$

- Consistent with eqn (5) of the previous Lecture, the **dynamic constraint** (2) can be expressed in **units of effective labour:**

$$k_{t+1}^{\#} \cdot (1+\mu_{N\#}) = f(k_t^{\#}) - c_t^{\#} + (1-\delta)k_t^{\#} \qquad (5)$$

II Optimal growth model

Solution based on Lagrange multiplier technique:

- To characterize the solution we optimize the objective

$$\begin{aligned}\mathcal{L} &= \sum_{t=0}^{\infty}\{[\beta\cdot(1+\mu_N)]^t\cdot\frac{((1+\mu_A)^t\cdot A_0\cdot c_t^{\#})^{1-\sigma}-1}{1-\sigma}\cdot N_0 \\ &+\lambda_t[f(k_t^{\#})-c_t^{\#}+(1-\delta)k_t^{\#}-k_{t+1}^{\#}\cdot(1+\mu_{N\#})]\}\end{aligned}$$

over the choice variables $\{c_t^{\#}, k_{t+1}^{\#}$, and $\lambda_t;\ \forall t\geqslant 0\}$, taking as given the predetermined value $k_0^{\#}$

II Optimal growth model

$$\mathcal{L} = \sum_{t=0}^{\infty}\{[\beta\cdot(1+\mu_N)]^t\cdot\frac{((1+\mu_A)^t\cdot A_0\cdot c_t^{\#})^{1-\sigma}-1}{1-\sigma}\cdot N_0 \\ +\lambda_t[f(k_t^{\#})-c_t^{\#}+(1-\delta)k_t^{\#}-k_{t+1}^{\#}\cdot(1+\mu_{N^{\#}})]\}$$

First-order optimality conditions ('FOCs', interior) w.r.t. $c_t^{\#}$, $k_t^{\#}$, λ_t:

$$\frac{\partial\mathcal{L}}{\partial c_t^{\#}} = [\beta\cdot(1+\mu_N)\cdot(1+\mu_A)^{1-\sigma}]^t\cdot A_0^{1-\sigma}\cdot N_0\cdot(c_t^{\#})^{-\sigma}-\lambda_t = 0 \quad t\geqslant 0 \quad (6)$$

$$\frac{\partial\mathcal{L}}{\partial k_t^{\#}} = \lambda_t[f'(k_t^{\#})+1-\delta]-\lambda_{t-1}\cdot(1+\mu_{N^{\#}}) = 0 \quad t>0 \quad (7)$$

$$\frac{\partial\mathcal{L}}{\partial \lambda_t} = f(k_t^{\#})-c_t^{\#}+(1-\delta)k_t^{\#}-k_{t+1}^{\#}\cdot(1+\mu_{N^{\#}}) = 0 \quad t\geqslant 0 \quad (8)$$

Transversality condition: $\lim_{t\to\infty}\lambda_t\cdot k_{t+1}^{\#} = 0$ (9)

II Optimal growth model

$\longrightarrow$ Combining eqns (6), (7) yields in 4 steps the **consumption Euler eqn**:

i) use eqn (6) to solve for λ_t and λ_{t-1}, respectively,

$$\begin{aligned}\lambda_t &= [\beta \cdot (1+\mu_N) \cdot (1+\mu_A)^{1-\sigma}]^t \cdot A_0^{1-\sigma} \cdot N_0 \cdot (c_t^{\#})^{-\sigma} \\ \lambda_{t-1} &= [\beta \cdot (1+\mu_N) \cdot (1+\mu_A)^{1-\sigma}]^{t-1} \cdot A_0^{1-\sigma} \cdot N_0 \cdot (c_{t-1}^{\#})^{-\sigma},\end{aligned}$$

ii) leading to

$$\frac{\lambda_t}{\lambda_{t-1}} = \beta \cdot (1+\mu_N) \cdot (1+\mu_A)^{1-\sigma} (\frac{c_t^{\#}}{c_{t-1}^{\#}})^{-\sigma}, \qquad (10)$$

iii) use this expression in eqn (7) to obtain

$$\beta \cdot (1+\mu_N) \cdot (1+\mu_A)^{1-\sigma} (\frac{c_t^{\#}}{c_{t-1}^{\#}})^{-\sigma} \cdot [f'(k_t^{\#}) + 1 - \delta] = 1 + \mu_{N^{\#}},$$

iv) divide the last eqn by $(1+\mu_N) \cdot (1+\mu_A) = 1 + \mu_{N^{\#}}$ to obtain

$$\beta \cdot (1+\mu_A)^{-\sigma} (\frac{c_t^{\#}}{c_{t-1}^{\#}})^{-\sigma} \cdot [f'(k_t^{\#}) + 1 - \delta] = 1 \qquad t > 0 \qquad (11)$$

II Optimal growth model

Consolidated intertemporal equilibrium:

→ The **consumption Euler equation** (11) and the **dynamic resource constraint** (8) give rise to the pair of **consolidated intertemporal equilibrium conditions:**

$$\beta \cdot (1+\mu_A)^{-\sigma} \left(\frac{c_{t+1}^{\#}}{c_t^{\#}}\right)^{-\sigma} \cdot [f'(k_{t+1}^{\#}) + 1 - \delta] = 1 \quad (12)$$

$$c_t^{\#} = f(k_t^{\#}) + (1-\delta)k_t^{\#} - (1+\mu_{N^{\#}}) \cdot k_{t+1}^{\#} \quad (13)$$

→ We have reduced the dynamics to a non-linear two-dimensional dynamic system in $c^{\#}$ and $k^{\#}$ with one initial condition ($k_0^{\#}$) and one terminal condition, as given by the transversality condition (11)

→ Notice: eqns (12) and (13) generalize eqns (16) and (17) in Lecture 2, allowing for $\mu_A > 0$ and $\mu_N > 0$

II Optimal growth model

Interpretation of the consumption Euler equation (12):

$\longrightarrow$ How does **consumption per capita** (ie c_t) evolve over time?

- Consider the consumption Euler equation (12), ie

$$\beta \cdot (1+\mu_A)^{-\sigma}\left(\frac{c^{\#}_{t+1}}{c^{\#}_t}\right)^{-\sigma} \cdot [f'(k^{\#}_{t+1}) + 1 - \delta] = 1$$

- Use $c^{\#}_t = \frac{c_t}{A_t}$ and $c^{\#}_{t+1} = \frac{c_{t+1}}{A_{t+1}}$ (which implies $\frac{c^{\#}_{t+1}}{c^{\#}_t} = \frac{c_{t+1}}{c_t}\frac{1}{1+\mu_A}$) to see that eqn (12)...:
  ...satisfies **in general** the well-known structure

$$U'(c_t) = \beta U'(c_{t+1})[f'(k^{\#}_{t+1}) + (1-\delta)]$$

  ...can for the **particular utility function** assumed above be rewritten as,

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \frac{f'(k^{\#}_{t+1}) + 1 - \delta}{1+\theta} \approx 1 + r_{t+1} - \theta, \qquad (14)$$

  using $\beta = \frac{1}{1+\theta}$ and $r_{t+1} = f'(k^{\#}_{t+1}) - \delta$.

II Optimal growth model

Interpretation of the consumption Euler equation (12):

$\rightarrow$ In view of eqn (14), ie

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} \approx 1 + r_{t+1} - \theta \quad \Leftrightarrow \frac{c_{t+1}}{c_t} \approx (1 + r_{t+1} - \theta)^{\frac{1}{\sigma}}$$

the behaviour of **consumption per capita** can be characterized as follows

Observation 1: c rises over time if $r_{t+1} > \theta$, ie if the interest rate exceeds the rate of time preference of HHs

Observation 2: The strength of the willingness to substitute consumption between periods depends on the intertemporal elasticity of substitution in consumption, as captured by $1/\sigma$

II Optimal growth model

Steady states of the equation system (12) and (13):

- In steady state the consumption Euler eqn (12) simplifies to

$$f'(k^{\#}) + 1 - \delta = \frac{1}{\beta} \cdot (1 + \mu_A)^{\sigma} = (1 + \theta) \cdot (1 + \mu_A)^{\sigma} \approx 1 + \theta + \sigma\mu_A,$$

  where the last step uses a first-order Taylor approximation around the values $\theta = \mu_A = 0$

- **In sum:** steady states of the system (12) and (13) satisfy

$$\begin{aligned} f'(k^{\#}) &\approx \delta + \theta + \sigma\mu_A && (15) \\ c^{\#} &= f(k^{\#}) - (\delta + \mu_{N^{\#}}) \cdot k^{\#} && (16) \end{aligned}$$

- These two equations have a recursive structure and are solved by a **unique steady state** with **solution values** $k^{\#*}$ **and** $c^{\#*}$

II Optimal growth model

Properties of the steady state:

- This unique steady–state solution generalizes the unique solution of eqns (18) and (19) of Lecture 2 to an environment with $\mu_A > 0$ and $\mu_N > 0$

- The two-dimensional dynamics in $c^\#$ and $k^\#$ with one initial condition ($k_0^\#$) and the terminal condition (11) are **saddlepath stable**, ie whenever $k_0^\# \neq k^{\#*}$ there exists a unique choice of the control variable $c_0^\#$ such that the economy 'jumps' on the saddlepath and converges over time towards the steady state $k^{\#*}$, $c^{\#*}$

II Optimal growth model

Similarities with the solution of the Solow-model:

- Corresponding to the steady-state values $c^{\#*}$ and $k^{\#*}$ (which are expressed in units of effective labour) there exists a unique balanced growth path. Along this path, per capita variables (ie k_t, y_t, c_t, i_t) grow at the constant rate μ_A

- Like in the Solow-model, the long-run growth rate of per capita variables μ_A is exogenous, ie not explained from within the model

- Like in the Solow-model, the implied differences in the per capita capital stock and return rates on capital are implausibly large to explain output per capita developments in a satisfactory manner

II Optimal growth model

Differences to the solution of the Solow-model:

- The **savings rate** is (only) during the transition period towards the steady state not constant
- The **steady-state value** $k^{\#*}$ is **always smaller** than the **golden-rule level**, ie
$$k^{\#*} < k^{\#}_{GR}$$
- **Why?** The assumed impatience of consumers implies that the long-run capital stock is not high enough to support the maximum level of steady-state consumption

II Optimal growth model

Background: Establishing $k^{\#*} < k^{\#}_{GR}$

$\rightarrow$ Combine the **golden-rule optimality criterion**

$$f'(k^{\#}_{GR}) = \delta + \mu_{N\#} \approx \delta + \mu_N + \mu_A$$

and the **optimality criterion (15)** of the **optimal growth model**

$$f'(k^{\#*}) \approx \delta + \theta + \sigma\mu_A$$

to establish

$$k^{\#*} < k^{\#}_{GR} \quad \Leftrightarrow \quad f'(k^{\#*}) > f'(k^{\#}_{GR}) \quad \Leftrightarrow \quad \theta - \mu_N - (1-\sigma)\mu_A > 0,$$

which is identical to the earlier imposed restriction (3)

II Optimal growth model

Background: Establishing $k^{\#*} < k^{\#}_{GR}$

- The restriction (3) can be linked to the **transversality condition (9),** ie
$$\lim_{t\to\infty} \lambda_t \cdot k^{\#}_{t+1} = 0$$
which must always be satisfied
- In particular, assume that $k^{\#}_{t+1}$ and $c^{\#}_t$ converge against the optimal long-run values $k^{\#*} > 0$, $c^{\#*} > 0$. Then, the TV-condition (9) will only be satisfied if
$$\lim_{t\to\infty} \lambda_t = 0$$
- Recall from eqn (10) that λ grows according to
$$\frac{\lambda_t}{\lambda_{t-1}} = \beta \cdot (1+\mu_N) \cdot (1+\mu_A)^{1-\sigma} (\frac{c^{\#}_t}{c^{\#}_{t-1}})^{-\sigma},$$
which, using $\beta = (1+\theta)^{-1}$, implies along a balanced growth path (with $c^{\#*}$ being constant)
$$\frac{\lambda_t}{\lambda_{t-1}} = \frac{(1+\mu_N) \cdot (1+\mu_A)^{1-\sigma}}{1+\theta}$$

II Optimal growth model

Background: Establishing $k^{\#*} < k^{\#}_{GR}$

- For $\lim_{t \to \infty} \lambda_t = 0$ we need $\frac{\lambda_t}{\lambda_{t-1}} < 1$, ie

$$\frac{\lambda_t}{\lambda_{t-1}} = \frac{(1+\mu_N)\cdot(1+\mu_A)^{1-\sigma}}{1+\theta} \approx 1 + \mu_N + (1-\sigma)\cdot\mu_A - \theta < 1,$$

which is equivalent to restriction (3), ie

$$\theta - \mu_N - (1-\sigma)\mu_A > 0$$

- **In sum**, for steady states to satisfy *all* optimality conditions the restriction (3) must hold. And this in turn implies $k^{\#*} < k^{\#}_{GR}$

II Optimal growth model

Comments: Steady-state properties of the optimal growth model

1) Since it is optimal to converge against the value $k^{\#*}$, this value is often called the **modified golden-rule capital stock**

2) The resulting pattern of

$$k^{\#*} < k^{\#}_{GR}$$

is called a constellation of **dynamic efficiency.** Such constellation is characterized by the feature that a reduction of the long-run capital stock does not increase long-run consumption.

Notice: As discussed above, the steady state of the Solow-model (ie $k^{\#}_{So}$) is not necessarily dynamically efficient.

II Optimal growth model

Background: Relaxing the objective of the Benthamite welfare function

- Assume one replaces the objective (1), ie $V_0 = \sum_{t=0}^{\infty} \beta^t U(c_t) \cdot N_t$, against the **alternative objective 'without population weights'**, ie

$$V_0 = \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{17}$$

- Notice that the objective (1) implied the effective discount factor

$$\frac{1+\mu_N}{1+\theta} \approx 1 + \mu_N - \theta$$

which under (17) will be replaced against the pure discount factor

$$\beta = \frac{1}{1+\theta} \approx 1 - \theta$$

- The resulting **stronger discounting** under the **alternative objective** (17) implies that the **long-run value of** $f'(k^{\#*})$ will be **higher**, ie

$$f'(k^{\#*}) \approx \delta + \theta + \mu_N + \sigma\mu_A,$$

making the **long-run value of the capital stock** $k^{\#*}$ **smaller**. Moreover, the modified condition (3) turns into

$$\theta - (1-\sigma)\mu_A > 0$$

III Endogenous growth
Ak-model

- Let us consider the most simple model available which derives the per capita growth rate of the economy endogenously, the so-called **Ak-model of endogenous growth**. The analysis goes back to Rebelo (1991)

- **Idea:** we maintain the assumption of **exogenous population growth** ($\mu_N > 0$), but there is **no labour-augmenting technological progress**

- Crucial mechanism for long-run per capita growth of output, consumption, and capital: the production function is such that **per capita output is linear in capital**

$$y_t = A \cdot k_t, \tag{18}$$

where $A > \delta + \theta > 0$ is a constant (and the raw labour input as previously captured by n_t is without return)

III Endogenous growth
Ak-model

Model ingredients:

- **Objective** (identical to eqn (1)):

$$V_0 = \sum_{t=0}^{\infty} \beta^t U(c_t) \cdot N_t$$

We will use again the particular utility function $U(c_t) = \frac{(c_t)^{1-\sigma}-1}{1-\sigma}$, implying

$$V_0 = \sum_{t=0}^{\infty} \beta^t \cdot \frac{(c_t)^{1-\sigma} - 1}{1-\sigma} \cdot N_t = \sum_{t=0}^{\infty} [\beta \cdot (1+\mu_N)]^t \cdot \frac{(c_t)^{1-\sigma} - 1}{1-\sigma} \cdot N_0 \qquad (19)$$

- **Dynamic constraint (per capita):**

$$k_{t+1} \cdot (1+\mu_N) = A \cdot k_t - c_t + (1-\delta) k_t, \qquad (20)$$

with the predetermined value k_0 to be taken as given

III Endogenous growth
Ak-model

- Can we expect per capita variables to grow over time or to be constant?
- At this stage, we don't know yet

$\longrightarrow$ If one solves the social planner's problem in terms of per capita variables the consolidated intertemporal equilibrium conditions look as follows:

Consumption Euler equation:

$$\beta \cdot \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} \cdot [A + 1 - \delta] = 1 \tag{21}$$

Resource constraint:

$$k_{t+1} \cdot (1 + \mu_N) = A \cdot k_t - c_t + (1 - \delta) k_t \tag{22}$$

III Endogenous growth
Ak-model

- Notice that the marginal product of capital (ie A) is constant
- Hence, the **consumption Euler equation,** ie (21)

$$\beta \cdot \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} \cdot [A + 1 - \delta] = 1$$

can be used to establish the **constant growth rate** of per **capita consumption** (ie $\mu_{\frac{C}{N}}$)

$$\frac{c_{t+1}}{c_t} = 1 + \mu_{\frac{C}{N}} = [\beta \cdot (A + 1 - \delta)]^{\frac{1}{\sigma}} = [\frac{A + 1 - \delta}{1 + \theta}]^{\frac{1}{\sigma}},$$

implying

$$\ln\left(\frac{c_{t+1}}{c_t}\right) = \frac{1}{\sigma} \cdot \ln\left(\frac{A + 1 - \delta}{1 + \theta}\right) \quad \Leftrightarrow \quad \mu_{\frac{C}{N}} \approx \frac{1}{\sigma} \cdot (A - \delta - \theta), \qquad (23)$$

where the approximation assumes that A, δ, and θ are numbers sufficiently close to zero

III Endogenous growth
Ak-model

Interpretation of eqn (23), ie

$$\mu_{\frac{C}{N}} \approx \frac{1}{\sigma} \cdot (A - \delta - \theta) \quad \Leftrightarrow \quad A - \delta \approx \theta + \sigma \mu_{\frac{C}{N}}$$

- In the Ak-model long-run per capita growth is feasible even if one assumes that there is no technological progress
- **Long-run growth** is driven by the model feature that **output moves in proportion to the capital stock**. This is different from both the original Solow-model and the optimal growth model, where it was assumed that $f(k)$ is subject to diminishing marginal productivity
- As a result, in the Ak-model the growth rate of per capita consumption depends on a range of model parameters (ie σ, A, δ, θ)
- *Notice:* The growth rate $\mu_{\frac{C}{N}}$ is independent of the starting position of the economy (no transitional dynamics)

III Endogenous growth
Ak-model

Interpretation of eqn (23), ie

$$\mu_{\frac{C}{N}} \approx \frac{1}{\sigma} \cdot (A - \delta - \theta) \quad \Leftrightarrow \quad A - \delta \approx \theta + \sigma \mu_{\frac{C}{N}}$$

$\rightarrow$ How does in the Ak-model the growth rate of per capita consumption depend on model parameters ?

Example 1: If the productivity of capital (ie A) increases this leads to an increase in $\mu_{\frac{C}{N}}$
Notice: A is assumed to be exogenous in the model, but one can imagine that policy interventions may have a chance to change A

Example 2: If consumers become more impatient (as captured by an increase in θ), this leads to a decline in $\mu_{\frac{C}{N}}$
Notice: Changes in θ are similar to changes in the savings behaviour. In particular, in the Solow-model, *when combined with an Ak-technology*, changes in s have long-run growth effects. Again, one can imagine that policy interventions may have a chance to change s

III Endogenous growth
Ak-model

Comparison between exogenous and endogenous growth:

→ For comparison, it is instructive to see that the long-run solutions as given by eqns (15) and (23) display a very similar structure:

$$f'(k^{\#*}) - \delta \approx \theta + \sigma \cdot \underbrace{\mu_{\frac{C}{N}}}_{=\mu_A} \qquad \text{vs.} \qquad \underbrace{f'(k)}_{=A} - \delta \approx \theta + \sigma \cdot \mu_{\frac{C}{N}}$$

Exogenous growth:

- The long-run per capita growth rate $\mu_{\frac{C}{N}}$ is uniquely fixed at the exogenous rate μ_A
- Changes in θ and σ leave $\mu_{\frac{C}{N}}$ unaffected and induce changes in $f'(k^{\#*})$

Endogenous growth (Ak model):

- The marginal product of capital is uniquely fixed at the value A
- Changes in θ and σ leave A unaffected and induce changes in the endogenous rate $\mu_{\frac{C}{N}}$

III Endogenous growth
Ak-model

Comparison between exogenous and endogenous growth:

$\longrightarrow$ These differences have significant implications for the quantitative predictions of the models, like whether we should expect **convergence of per capita incomes** or not

Example: consider 2 countries with different attitudes towards intertemporal consumption as captured by θ

$\longrightarrow$ **Conditional convergence** if long-run per capita growth is **exogenous**
$\longrightarrow$ **No convergence** if long-run per capita growth is **endogenous** in the spirit of the Ak-model

Notice: Changes in θ are similar to changes in the savings behaviour. In particular, in the Solow-model, *when combined with an Ak-technology*, 2 countries with different savings rates display different long-run growth rates, ie under an *Ak-technology* there is no convergence (absolute or conditional) of per capita incomes

III Endogenous growth
Ak-model

How should one read the Ak-model?

→ Essentially, the Ak-model offers a short-cut in reduced form which shows that output can grow in proportion to capital if there exists some mechanism which offsets the standard assumption of diminishing private returns to physical capital

→ To conclude the lecture, we show that such mechanism - under two alternative and more nuanced modelling approaches - can be made consistent with:

1) the notion of **constant returns to scale in the aggregate production function with respect to physical capital and labour**
(→ Frankel, 1962, Romer, 1986)

2) the idea of a **broad measure of capital which allows for both physical and human capital**
(→ Lucas, 1988)

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Re-interpreting the Ak-model: externalities in capital accumulation

→ For many macroeconomic applications, a convenient starting point is the assumption of an aggregate production function in capital and labour of the Cobb-Douglas-type, ie

$$Y = F(K, N) = A \cdot K^{\alpha} N^{\gamma}$$

To reconcile this assumption with the Ak-feature of a linear relationship between output and capital is a priori not straightforward. Why?

→ The Ak-feature requires $\alpha = 1$. This implies:

1) Labour can only play an active role (ie $\gamma > 0$) if one replaces the assumption of constant returns to scale ('CRTS') in K and L against the assumption of increasing returns to scale (IRTS). But this is difficult to reconcile with the existence of a competitive equilibrium in a decentralized environment in which firms maximize profits, taking factor prices of capital and labour as given

2) Conversely, the assumption of CRTS can only be maintained in the degenerate scenario in which (raw) labour is entirely unproductive ($\gamma = 0$)

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Re-interpreting the Ak-model: externalities in capital accumulation

Frankel (1962) and Romer (1986):

- In general A stands for the knowledge embodied in the economy-wide capital stock...
- ...and the constancy of A may well capture appropriately sized externalities associated with the investment decisions of firms

$\rightarrow$ The i) notion of CRTS in K and L in the aggregate production function and ii) the Ak-feature of a linear relationship between capital and output can be reconciled

Idea:

- individual firms perceive diminishing private returns to capital at the firm level
- at the aggregate level, positive externalities of investment decisions generate constant social returns to capital
- all this is consistent with CRTS between labour and capital in the aggregate production function

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Re-interpreting the Ak-model: externalities in capital accumulation

The main ingredients of Romer (1986) can be adapted to the above sketched Ak-model as follows:

- Aggregate output is produced according to the technology:

$$Y_t = \overline{A}_t \cdot K_t^{\alpha} N_t^{1-\alpha},$$

- Per capita output:

$$y_t = \frac{Y_t}{N_t} = \overline{A}_t \cdot k_t^{\alpha}$$

- The variable $\overline{A}_t$ captures an externality which is a function of the capital-labour ratio such that

$$\overline{A}_t = A \cdot k_t^{\beta}$$

- Consider the special parameter case $\beta = 1 - \alpha$. Then, the externality exactly offsets the diminishing private returns to capital induced by α such that per capita output becomes

$$y_t = A \cdot k_t$$

- **Crucial:** the social planner solution internalizes the externality associated with $\overline{A}_t$, differently from the solution provided by decentralized markets.

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Re-interpreting the Ak-model: externalities in capital accumulation

Problem of the social planner:

- Objective ($\longrightarrow$ **identical to eqn (19)**):

$$V_0 = \sum_{t=0}^{\infty} \beta^t U(c_t) \cdot N_t = \sum_{t=0}^{\infty} [\beta \cdot (1+\mu_N)]^t \cdot \frac{(c_t)^{1-\sigma} - 1}{1-\sigma} \cdot N_0$$

- Dynamic constraint in per capita terms **($\longrightarrow$ identical to eqn (20)):**

$$k_{t+1} \cdot (1+\mu_N) = A \cdot k_t - c_t + (1-\delta) k_t,$$

- Consumption Euler equation **($\longrightarrow$ identical to eqn (21)):**

$$\beta \cdot \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} \cdot [A + 1 - \delta] = 1$$

- **Socially optimal growth rate of consumption per capita ($\longrightarrow$ identical to eqn (23)):**

$$\ln\left(\frac{c_{t+1}}{c_t}\right) = \frac{1}{\sigma} \cdot \ln\left(\frac{A+1-\delta}{1+\theta}\right) \quad \Leftrightarrow \quad \mu_{\frac{C}{N}, SP} \approx \frac{1}{\sigma} \cdot (A - \delta - \theta), \qquad (24)$$

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Re-interpreting the Ak-model: externalities in capital accumulation

Solution provided by decentralized markets:

$\rightarrow$ Representative consumer/owner of the representative firm faces the same maximization problem, but he or she takes as given the process $\overline{A}_t$

- Modified dynamic constraint in per capita terms (ie the representative firm does not perceive a linear technology):

$$k_{t+1} \cdot (1 + \mu_N) = \overline{A}_t \cdot k_t^{\alpha} - c_t + (1 - \delta) k_t$$

- Modified consumption Euler equation:

$$\beta \cdot (\frac{c_{t+1}}{c_t})^{-\sigma} \cdot [\alpha \cdot \overline{A}_t \cdot k_t^{\alpha - 1} + 1 - \delta] = 1$$

- The competitive equilibrium satisfies

$$\overline{A}_t = A \cdot k_t^{1-\alpha},$$

  implying for the consumption Euler equation

$$\beta \cdot (\frac{c_{t+1}}{c_t})^{-\sigma} \cdot [\alpha \cdot A + 1 - \delta] = 1$$

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Re-interpreting the Ak-model: externalities in capital accumulation

Solution provided by decentralized markets:

- Thus, the growth rate of consumption per capita generated by the decentralized market solution is given by

$$\mu_{\frac{C}{N},\,MS} \approx \frac{1}{\sigma} \cdot (\alpha \cdot A - \delta - \theta) \tag{25}$$

- Comparing (24) and (25) yields:

$$\underbrace{\frac{1}{\sigma} \cdot (\alpha \cdot A - \delta - \theta)}_{\mu_{\frac{C}{N},\,MS}} < \underbrace{\frac{1}{\sigma} \cdot (A - \delta - \theta)}_{\mu_{\frac{C}{N},\,SP}}$$

Interpretation:

- In the market solution individuals fail to internalize the effect of individual capital accumulation on knowledge $\overline{A}_t$
- As a result, the equilibrium growth rate of per capita variables is less than the socially optimal one

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Re-interpreting the Ak-model: externalities in capital accumulation

Comment:

- The process driving the externality $\overline{A}_t = A \cdot k_t^{\beta}$ is a general one, while the assumed parameter constellation

  $$\beta = 1 - \alpha$$

  which reproduces the Ak-model is very special

- If one assumes, alternatively,

  $$\beta < 1 - \alpha$$

  per capita growth vanishes in the long run in the absence of exogenous technological progress (similar to the Solow model and the optimal growth model)

- In sum, while this alternative assumption makes the long-run growth predictions of the model close to the Solow model, the elasticity of output with respect to (economy-wide) capital increases, improving on the ability of Solow-type models, when augmented with externalities, to predict cross-country differences in $\frac{Y}{N}$ via differences in $\frac{K}{N}$

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Re-interpreting the Ak-model: the role of human capital

$\longrightarrow$ A second and alternative mechanism to re-interpret the Ak-model as a reduced form of a model with more compelling micro-foundations relates to the idea of a **broad measure of capital which allows for separate contributions to output from physical and human capital**

$\longrightarrow$ For illustration, we consider a much simplified version of the model provided by Lucas (1988)

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Re-interpreting the Ak-model: the role of human capital

Model ingredients:

- We maintain the assumption of exogenous population growth ($\mu_N > 0$), but there is **no labour-augmenting technological progress**
- The production function has CRTS w.r.t. to the two inputs **physical capital** (K) and **human capital** (H)
  (and raw labour N plays no role)
- Per capita production function:

$$y_t = A \cdot k_t^{\alpha} \cdot h_t^{1-\alpha} \qquad \text{with:} \quad \alpha \in (0,1)$$

- Both types of capital subject to the same rate of depreciation $\delta \in (0,1)$

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Re-interpreting the Ak-model: the role of human capital

Model ingredients:

- **Objective** ($\longrightarrow$ identical to eqn (19)):

$$V_0 = \sum_{t=0}^{\infty} \beta^t U(c_t) \cdot N_t = \sum_{t=0}^{\infty} [\beta \cdot (1+\mu_N)]^t \cdot \frac{(c_t)^{1-\sigma} - 1}{1-\sigma} \cdot N_0$$

- **Dynamic constraint (per capita):**

$$(k_{t+1} + h_{t+1}) \cdot (1+\mu_N) = A \cdot k_t^{\alpha} \cdot h_t^{1-\alpha} - c_t + (1-\delta)(k_t + h_t),$$

with the predetermined values k_0 and h_0 to be taken as given

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Re-interpreting the Ak-model: the role of human capital

Solution based on Lagrange multiplier technique:

To characterize the solution we optimize the objective

$$\mathcal{L} = \sum_{t=0}^{\infty} \{[\beta \cdot (1+\mu_N)]^t \cdot \frac{(c_t)^{1-\sigma}-1}{1-\sigma} \cdot N_0$$

$$+\lambda_t [A \cdot k_t^{\alpha} \cdot h_t^{1-\alpha} - c_t + (1-\delta)(k_t + h_t) - (k_{t+1} + h_{t+1}) \cdot (1+\mu_N)]\}$$

over the choice variables $\{c_t, k_{t+1}, h_{t+1}$ and $\lambda_t; \forall t \geqslant 0\}$, taking as given the predetermined values k_0 and h_0

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$$\mathcal{L} = \sum_{t=0}^{\infty} \{[\beta \cdot (1+\mu_N)]^t \cdot \frac{(c_t)^{1-\sigma} - 1}{1-\sigma} \cdot N_0$$
$$+\lambda_t [A \cdot k_t^{\alpha} \cdot h_t^{1-\alpha} - c_t + (1-\delta)(k_t + h_t) - (k_{t+1} + h_{t+1}) \cdot (1+\mu_N)]\}$$

First-order optimality conditions (interior) w.r.t. c_t, k_t, h_t and λ_t :

$$\frac{\partial \mathcal{L}}{\partial c_t} = [\beta \cdot (1+\mu_N)]^t \cdot N_0 \cdot (c_t)^{-\sigma} - \lambda_t = 0 \quad t \geqslant 0$$

$$\frac{\partial \mathcal{L}}{\partial k_t} = \lambda_t [\alpha \cdot A \cdot k_t^{\alpha-1} \cdot h_t^{1-\alpha} + 1 - \delta] - \lambda_{t-1} \cdot (1+\mu_N) = 0 \quad t > 0$$

$$\frac{\partial \mathcal{L}}{\partial h_t} = \lambda_t [(1-\alpha) \cdot A \cdot k_t^{\alpha} \cdot h_t^{-\alpha} + 1 - \delta] - \lambda_{t-1} \cdot (1+\mu_N) = 0 \quad t > 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = A \cdot k_t^{\alpha} \cdot h_t^{1-\alpha} - c_t + (1-\delta)(k_t + h_t)$$
$$-(k_{t+1} + h_{t+1}) \cdot (1+\mu_N) = 0 \qquad t \geqslant 0$$

Transversality conditions: $\lim_{t\to\infty} \lambda_t \cdot k_{t+1} = 0$ and $\lim_{t\to\infty} \lambda_t \cdot h_{t+1} = 0$

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Re-interpreting the Ak-model: the role of human capital

→ Because of the symmetric structure of the production function with respect to k and h, we get the pair of Euler equations:

$$\begin{aligned} \beta \cdot \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} \cdot [\alpha \cdot A \cdot k_{t+1}^{\alpha-1} \cdot h_{t+1}^{1-\alpha} + 1 - \delta] &= 1 \\ \beta \cdot \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} \cdot [(1-\alpha) \cdot A \cdot k_{t+1}^{\alpha} \cdot h_{t+1}^{-\alpha} + 1 - \delta] &= 1 \end{aligned}$$

→ These two equations imply

$$\frac{k_{t+1}}{h_{t+1}} = \frac{k}{h} = \frac{\alpha}{1-\alpha}, \tag{26}$$

ie the **ratio between physical and human capital is constant**

→ Thus, we can simplify the above pair of equations to the single **consumption Euler equation**

$$\beta \cdot \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} \cdot [\alpha \cdot A \cdot \left(\frac{k}{h}\right)^{\alpha-1} + 1 - \delta] = 1 \tag{27}$$

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Re-interpreting the Ak-model: the role of human capital

Interpretation of the consumption Euler equation (27), ie:

$$\beta \cdot (\frac{c_{t+1}}{c_t})^{-\sigma} \cdot [\alpha \cdot A \cdot (\underbrace{\frac{k}{h}}_{\frac{\alpha}{1-\alpha}})^{\alpha-1} + 1 - \delta] = 1$$

- *Notice*: since $\frac{k}{h}$ is constant, eqn (27) has a structure which is identical to the consumption Euler eqn (21) of the Ak-model, ie capital is subject to a constant marginal product along the optimal growth path (and there are no transitional dynamics)
- This finding can also be seen from substituting (26) into the production function

  $$y_t = A \cdot k_t^{\alpha} \cdot h_t^{1-\alpha} = A \cdot (\frac{k_t}{h_t})^{\alpha-1} \cdot k_t = \widetilde{A} \cdot k_t,$$

  ie we obtain the Ak-production function with

  $$\widetilde{A} = A \cdot (\frac{\alpha}{1-\alpha})^{\alpha-1}$$

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Re-interpreting the Ak-model: the role of human capital

Interpretation of the consumption Euler equation (27), ie:

- Accordingly, the **growth rate** of per **capita consumption** ($\mu_{\frac{C}{N}}$) is constant, ie

$$\frac{c_{t+1}}{c_t} = 1 + \mu_{\frac{C}{N}} = [\beta \cdot (\alpha \cdot \widetilde{A} + 1 - \delta)]^{\frac{1}{\sigma}} = [\frac{\alpha \cdot \widetilde{A} + 1 - \delta}{1 + \theta}]^{\frac{1}{\sigma}},$$

  implying

$$\mu_{\frac{C}{N}} \approx \frac{1}{\sigma} \cdot (\alpha \cdot \widetilde{A} - \delta - \theta),$$

  where the approximation assumes that A, δ, and θ are numbers sufficiently close to zero.

- **In sum:** it is possible to re-interpret the Ak-model as a reduced form of a model which allows for constant returns to scale to the 2 inputs physical capital and human capital
- **Comment:** Lucas (1988) has a model which is richer. In particular, his model allows for externalities associated with the formation of human capital. This implies that the solution of the social planner and the solution provided by decentralized markets are not identical