Lecture 5
Economic Growth: Foundations

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I Motivation

This Lecture considers extensions of the basic model that was discussed in Lecture 2 and addresses various aspects of **economic growth**

Goal:

- The Lecture focuses on selected theoretical and empirical issues
- It starts out with the basic growth model of Solow (1956) and paves the way for the next Lecture which gives an introduction to various questions addressed in modern growth theory

Most of the notation (including the discrete time treatment of growth issues) is in line with *Wickens, Chapter 3*, while in terms of substance this lecture draws extensively on the more detailed exposition of growth issues in: *Romer, D., Advanced Macroeconomics, 3rd edition, McGraw-Hill, 2006, Chapters 1 and 3*

Moreover, see:
→ *Aghion, P. and Howitt, P., Endogenous Growth Theory, MIT Press, 1998, Chapter 1*
→ *Lucas, R., On the Mechanics of Economic Development, Journal of Monetary Economics, 22, 3-42, 1988*

I Motivation

Economic growth is a fascinating and broad topic because of the richness of the empirical facts that seek explanation

Some dimensions of the data:

1) Over time, average real per capita incomes in industrialized countries like the US or in Europe have risen significantly

- Today they are about 10-30 times larger than a century ago
- Average growth rates in the 20th century in these countries were higher than in the 19th century which were in turn higher than in the 18th century
- On the eve of the Industrial Revolution average incomes in the wealthiest countries were not significantly above subsistence levels

I Motivation

2) Between countries average real per capita incomes show very significant dispersion

- Per capita incomes in the rich countries exceed those of the poor countries at least by a factor of 10 to 20, and in some cases differences may be much larger (but measurement issues are significant)
- The variability of growth patterns between countries is striking:

→ in some countries, like Argentina, average growth rates declined over many decades, despite a very strong performance of the country in the early phase of development
→ other countries, like a number of Sub-Saharan countries, never succeeded to obtain sustained increases in per capita incomes
→ by contrast, countries like Japan or the NIC's of East Asia managed to create growth miracles within relatively short periods of time (ie since the NIC's took off in the 1960s their per capita incomes relative to the US have more than tripled)

I Motivation

Moreover, the **implications of seemingly small changes in growth rates for standards of living** are staggering

→ **Lucas (1988)** in a key contribution to modern growth theory summarizes striking features of the **arithmetic of growth rates** as follows:

"Rates of growth of real per capita GNP are...diverse, even over sustained periods. For 1960-80 we observe, for example, India, 1.4% per year; Egypt, 3.4%; South Korea, 7.0%; Japan, 7.1%, the United States, 2.3%, the industrial economies averaged 3.6%." (Lucas, 1988, p.3)

Using these numbers, Lucas illustrates their implications as follows:

"...Indian incomes will double every 50 years; Korean every 10. An Indian will, on average, be twice as well off as his grandfather; a Korean 32 times."

I Motivation

Lucas (1988) then moves on to motivate his perspective on modern growth theory as follows:

"I do not see how one can look at figures like these without seeing them as representing possibilities. Is there some action a government of India could take that would lead the Indian economy to grow like...Egypt's? If so, what, exactly? If not, what is about the 'nature of India' that makes it so? The consequences for human welfare involved in questions like these are simply staggering: Once one starts to think about them, it is hard to think about anything else.
This is what we need a theory of economic development for: to provide some kind of framework for organizing facts like these, for judging which represent opportunities and which necessities."

Comment:
→ More than 20 years later we better treat some of the country names in Lucas' quote as placeholders!
→ Title story of the Economist, October 2, 2010: **How India's growth will outpace China's**

II Solow model: Theory
Model ingredients

The model by Solow (1956) extends the benchmark aggregate production function

$$Y_t = F(K_t, N)$$

in two dimensions

- First, the model assumes constant and exogenous **population growth** over time , ie

$$N_{t+1} = (1 + \mu_N) \cdot N_t, \quad \text{with: } \mu_N > 0 \tag{1}$$

Notice: since it is assumed that per capita labour supply is inelastically fixed at $n = 1$, this assumption is equivalent to assuming that the labour force (ie the aggregate labour supply) grows at the constant rate $\mu_N > 0$

- Second, to capture progress in knowledge or the effectiveness of labour, the model assumes constant and **exogenous labour augmenting technological progress**, ie

$$A_{t+1} = (1 + \mu_A) \cdot A_t, \quad \text{with: } \mu_A > 0 \tag{2}$$

II Solow model: Theory

Model ingredients

- Let us define the level of **'effective labour'** in period t as

$$N_t^{\#} = A_t \cdot N_t$$

with

$$N_{t+1}^{\#} = A_{t+1} \cdot N_{t+1} = (1 + \mu_A) \cdot (1 + \mu_N) \cdot N_t^{\#}$$

- Similarly, we define the constant **growth rate** of **effective labour** as

$$(1 + \mu_{N^{\#}}) = (1 + \mu_A) \cdot (1 + \mu_N), \quad \text{with: } \mu_{N^{\#}} > 0$$

$\rightarrow$ Notice: for **small** values of μ_A and μ_N, one can use approximately

$$\mu_{N^{\#}} \approx \mu_A + \mu_N$$

- This leads to the modified aggregate **production function**

$$Y_t = F(K_t,\ A_t \cdot N_t) = F(K_t, N_t^{\#}),$$

where aggregate output results from the combination of the two inputs 'capital' and 'effective labour'

II Solow model: Theory

Model ingredients

- The aggregate production function

$$Y_t = F(K_t, N_t^{\#})$$

is subject to **constant returns to scale**

- It will be convenient to express most variables in **intensive form**, ie we will consider Y_t, K_t, C_t, I_t etc. **per unit of effective labour**:

$$y_t^{\#} = \frac{Y_t}{N_t^{\#}}, \quad k_t^{\#} = \frac{K_t}{N_t^{\#}}, \quad c_t^{\#} = \frac{C_t}{N_t^{\#}}, \quad i_t^{\#} = \frac{I_t}{N_t^{\#}}$$

- Because of constant returns to scale, output in intensive form satisfies

$$y_t^{\#} = \frac{F(K_t, N_t^{\#})}{N_t^{\#}} = \frac{N_t^{\#} \cdot F(\frac{K_t}{N_t^{\#}}, 1)}{N_t^{\#}} = F(k_t^{\#}, 1) \equiv f(k_t^{\#}),$$

and the intensive form production function $f(k^{\#})$ is assumed to satisfy the **neoclassical properties** discussed earlier, ie for $k^{\#} > 0$:

$$\begin{aligned} f(k^{\#}) &> 0,\ f'(k^{\#}) > 0, f''(k^{\#}) < 0 \\ \lim_{k^{\#} \to 0} f'(k^{\#}) &\to \infty,\ \lim_{k^{\#} \to \infty} f'(k^{\#}) \to 0 \end{aligned}$$

II Solow model: Theory

Model ingredients

Remarks on growth rates:

- For illustration, consider the law of motion of labour augmenting technological progress (eqn (2)), ie

$$A_{t+1} = (1 + \mu_A) \cdot A_t, \quad \text{with: } \mu_A > 0$$

The parameter μ_A denotes the growth rate of A_t since

$$\frac{\Delta A_{t+1}}{A_t} = \frac{A_{t+1} - A_t}{A_t} = \frac{(1 + \mu_A) \cdot A_t - A_t}{A_t} = \mu_A$$

- Notice that it is often convenient to derive the (approximate) growth rates of variables from the log of their ratios, ie

$$\ln\left(\frac{A_{t+1}}{A_t}\right) = \ln(A_{t+1}) - \ln(A_t) = \ln(1 + \mu_A) \approx \mu_A,$$

where the approximation is very accurate for small values of μ_A
(and the literature often ignores the qualification 'approximately equal')

II Solow model: Theory

Model ingredients

Normalization of starting values:

- From eqns (1) and (2), the exogenous laws of motions of N_t and A_t are given by

$$\begin{aligned} N_t &= (1+\mu_N)^t \cdot N_0, \\ A_t &= (1+\mu_A)^t \cdot A_0 \end{aligned}$$

- Similarly

$$N_t^{\#} = (1+\mu_{N^{\#}})^t \cdot N_0^{\#} = (1+\mu_{N^{\#}})^t \cdot A_0 N_0$$

- Often it is convenient to normalize the starting values as $N_0 = A_0 = 1$
- *Implication:* when comparing countries with identical growth rates of μ_N and μ_A, this normalization implies that possible cross-country differences in the levels of variables like output or consumption cannot be due to differences in the starting values N_0 and A_0

II Solow model: Theory
Model ingredients

Remarks on the nature of technological progress:

- The assumption of **labour-augmenting technological progress** has a number of empirically appealing implications for the **long-run** solution of the model:
  $\longrightarrow$ the capital-output ratio will be constant
  $\longrightarrow$ the real interest rate will be trendless, while the real wage rate will grow over time
  $\longrightarrow$ in per capita terms, output and capital will grow over time, while the labour supply is trendless
  $\longrightarrow$ the income shares of capital and labour will be constant

II Solow model: Theory

Model ingredients

Remarks on the nature of technological progress:

- At the outset, it may have been more intuitive to consider **factor-neutral (**or **Hicks-neutral) technological progress**, ie

$$Y_t = \widetilde{A}_t \cdot F(K_t, N_t),$$

with

$$\widetilde{A}_{t+1} = (1 + \mu_{\widetilde{A}}) \cdot \widetilde{A}_t, \quad \text{with: } \mu_{\widetilde{A}} > 0$$

- This alternative assumption is called neutral since it implies that **output growth at constant inputs** ($K_t = K$, $N_t = N$) would be given by

$$\ln(\frac{Y_{t+1}}{Y_t}) = \ln(Y_{t+1}) - \ln(Y_t) = \ln(1 + \mu_{\widetilde{A}}) \approx \mu_{\widetilde{A}}$$

II Solow model: Theory
Model ingredients

Remarks on the nature of technological progress:

- The **Cobb-Douglas function** offers a direct **link** between **labour augmenting** progress **(A_t)** and **factor-neutral** progress **($\widetilde{A}_t$)**

$$Y_t = K_t^{\alpha} \cdot (N_t^{\#})^{1-\alpha} = K_t^{\alpha} \cdot (A_t N_t)^{1-\alpha} = \underbrace{A_t^{1-\alpha}}_{\widetilde{A}_t} \cdot K_t^{\alpha} \cdot N_t^{1-\alpha}$$

- Using $\widetilde{A}_t = A_t^{1-\alpha}$, this leads to a link between the respective growth rates:

$$\ln\left(\frac{\widetilde{A}_{t+1}}{\widetilde{A}_t}\right) = \ln(A_{t+1}^{1-\alpha}) - \ln(A_t^{1-\alpha}) = (1-\alpha)\ln(1+\mu_A) \approx (1-\alpha)\mu_A,$$

implying

$$\mu_A \approx \frac{\mu_{\widetilde{A}}}{1-\alpha}$$

II Solow model: Theory

Model ingredients

Overview: 3 different representations of endogenous variables

- **Aggregate variables** Y, K, C, I etc. are subject to growth from population growth and technological progress
- **Per capita variables** y, k, c, i etc. are subject to growth from technological progress
- **Variables in terms of effective labour** $y^{\#}$, $k^{\#}$, $c^{\#}$, $i^{\#}$ etc. have a chance to be constant over time

- **Example:** Cobb-Douglas production function

Aggregate output: $Y_t = K_t^{\alpha} \cdot (N_t^{\#})^{1-\alpha} = K_t^{\alpha} \cdot (A_t N_t)^{1-\alpha}$

Output per capita: $\dfrac{Y_t}{N_t} = y_t = A_t^{1-\alpha} \cdot k_t^{\alpha}$

Output per unit of effective labour : $\dfrac{Y_t}{N_t^{\#}} = y_t^{\#} = (k_t^{\#})^{\alpha}$

II Solow model: Theory

Dynamics of the model

- In aggregate terms, let us combine the **national income identity**

$$Y_t = C_t + I_t,$$

the equation defining the **capital stock dynamics**

$$\Delta K_{t+1} = I_t - \delta K_t,$$

and the **production function**

$$Y_t = F(K_t, N_t^{\#})$$

to get the dynamic equation

$$K_{t+1} = F(K_t, N_t^{\#}) - C_t + (1-\delta)K_t \tag{3}$$

- **Key behavioural assumption** of the Solow-model: savings S_t are a constant fraction of income (ie the **savings rate** s_t is **constant**):

$$s_t = \frac{S_t}{Y_t} = \frac{Y_t - C_t}{Y_t} = s, \quad \text{with: } s \in (0,1) \tag{4}$$

II Solow model: Theory

Dynamics of the model

- Let us combine equations (3) and (4), ie

$$K_{t+1} = F(K_t, N_t^{\#}) - C_t + (1-\delta)K_t \quad \text{and} \quad s_t = \frac{Y_t - C_t}{Y_t} = s$$

to establish the **law of motion**...

- ... of the **aggregate capital stock**

$$K_{t+1} = s \cdot F(K_t, N_t^{\#}) + (1-\delta)K_t$$

- ... of the **capital stock** in **per capita** form

$$k_{t+1} \cdot (1+\mu_N) = s \cdot \frac{F(K_t, N_t^{\#})}{N_t} + (1-\delta)k_t,$$

using $\frac{K_{t+1}}{N_t} = \frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} = k_{t+1} \cdot (1+\mu_N)$

- ...of the **capital stock** per unit of **effective labour**

$$k_{t+1}^{\#} \cdot (1+\mu_{N^{\#}}) = s \cdot f(k_t^{\#}) + (1-\delta)k_t^{\#}, \tag{5}$$

using $\frac{K_{t+1}}{N_t^{\#}} = \frac{K_{t+1}}{N_{t+1}^{\#}} \frac{N_{t+1}^{\#}}{N_t^{\#}} = k_{t+1}^{\#} \cdot (1+\mu_{N^{\#}})$

II Solow model: Theory

Dynamics of the model

Eqn (5), ie the **law of motion** of the **capital stock** per unit of **effective labour**

$$k^{\#}_{t+1} \cdot (1 + \mu_{N^{\#}}) = s \cdot f(k^{\#}_t) + (1 - \delta) k^{\#}_t,$$

is the **central equation** of the Solow model

Interpretation:

- The RHS denotes per unit of effective labour the period-t contribution to the capital stock available in period $t+1$
- To convert this into period-$t+1$ units one needs to **adjust** for both **population growth** and **technological progress**, ie one obtains only

  $$\frac{s \cdot f(k^{\#}_t) + (1 - \delta) k^{\#}_t}{(1 + \mu_A) \cdot (1 + \mu_N)} = \frac{s \cdot f(k^{\#}_t) + (1 - \delta) k^{\#}_t}{1 + \mu_{N^{\#}}}$$

  units of capital per effective labour in period $t+1$

II Solow model: Theory

Steady-state solution

- The steady-state version of eqn (5) satisfies $k_t^{\#} = k^{\#}$ and is given by

$$s \cdot f(k^{\#}) = (\delta + \mu_{N^{\#}})k^{\#} \tag{6}$$

Interpretation:

- The LHS of (6) measures the gross investment per unit of effective labour
- The RHS of (6) describes the extra or **break-even investment** that is needed to keep $k^{\#}$ constant at its existing level.
  It includes the replacement of depreciated capital as well as compensations that are needed to catch up with population growth and technological progress (both contained in $\mu_{N^{\#}}$)

II Solow model: Theory
Steady-state solution

Eqn (6), ie

$$s \cdot f(k^{\#}) = (\delta + \mu_{N^{\#}})k^{\#}$$

defines implicitly a **unique steady-state solution** $k^{\#}_{So}$, *assuming* $k^{\#} > 0$

Comment: Uniqueness of the steady state

- Most neoclassical production functions (including the Cobb-Douglas function) satisfy

  $$f(0) = 0,$$

  implying that there exists a second and degenerate solution of eqn (6) with $k^{\#} = 0$
- In the following, this solution will be ignored since it is unstable and of little economic interest
- Moreover, not all neoclassical production functions give rise to such second solution (*example:* CES production function with elasticity of substitution larger than 1)

II Solow model: Theory

Steady-state solution

Stability of the steady state:

- Consider again the central equation (5), ie

$$k^{\#}_{t+1} \cdot (1 + \mu_{N^{\#}}) = s \cdot f(k^{\#}_t) + (1 - \delta) k^{\#}_t,$$

  which is a **non-linear first-order difference equation in** $k^{\#}$

- If one rewrites eqn (5) as

$$\Delta k^{\#}_{t+1} = \frac{s \cdot f(k^{\#}_t) - (\delta + \mu_{N^{\#}}) k^{\#}_t}{1 + \mu_{N^{\#}}}$$

  it is easy to confirm from the corresponding **phase-diagram** that the **steady state** $k^{\#}_{So}$ is **globally stable**, ie for initial values $k^{\#}_t \neq k^{\#}_{So}$ (and $k^{\#}_t \neq 0$) the economy converges over time against the value $k^{\#}_{So}$

II Solow model: Theory
Steady-state solution

Interpretation of the steady-state solution as a balanced growth path:

- What does it mean for the other variables of the economy when $k^{\#}$ equals $k^{\#}_{So}$?
- Recall that A_t and N_t grow at the **exogenous** rates μ_A and μ_N

Aggregate variables:

- Consider the behaviour of the aggregate capital stock $K_t = A_t \cdot N_t \cdot k^{\#}_t$
- In **steady state**, $k^{\#}_t$ is constant (ie $k^{\#}_t = k^{\#}_{So}$), implying that K_t grows at the constant rate $\mu_{N\#}$ (and recall: $\mu_{N\#} \approx \mu_A + \mu_N$)
- Similarly, in **steady state**

$$\begin{aligned} y^{\#}_{So} &= f(k^{\#}_{So}) \\ i^{\#}_{So} &= sf(k^{\#}_{So}) \\ c^{\#}_{So} &= f(k^{\#}_{So}) - (\delta + \mu_{N\#}) \cdot k^{\#}_{So} \end{aligned} \tag{7}$$

implying that Y_t, I_t, and C_t also grow at the rate $\mu_{N\#}$

II Solow model: Theory

Steady-state solution

Interpretation of the steady-state solution as a balanced growth path:

Per capita variables

- Consider the behaviour of the capital stock in per capita terms $k_t = A_t \cdot k_t^{\#}$
- In steady state, $k_t^{\#}$ is constant (ie $k_t^{\#} = k_{So}^{\#}$), implying that k_t grows at the constant rate μ_A
- Accordingly, y_t, i_t, and c_t also grow at the rate μ_A

II Solow model: Theory

Steady-state solution

Interpretation of the steady-state solution as a balanced growth path:

In sum: the Solow model implies that, as $k_t^{\#}$ converges against $k_{So}^{\#}$, the economy converges against a **balanced growth path**

- The existence of a balanced growth path describes a situation where all variables grow at constant rates
- In particular, in the Solow model growth rates of per capita variables are solely determined by the rate of technological progress
- In other words: permanent changes to the determinants of $k_{So}^{\#}$ (like the savings rate s or the share parameter α) affect only the levels of variables, but not their long-run growth rates

II Solow model: Theory

Effects of a change in the savings rate

Effect of a permanent change in the savings rate s on output:

- Assume that the savings rate s increases permanently, possibly because of a gov't intervention which makes it more rewarding to save. What is the effect of such change?
- Consider the steady-state eqn (6). Since in steady state this eqn will be satisfied for any value of the parameter s, we can write it as an identity, using the implicit relationship $k^{\#}_{So} = k^{\#}_{So}(s)$:

$$s \cdot f(k^{\#}_{So}(s)) = (\delta + \mu_{N^{\#}}) \cdot k^{\#}_{So}(s) \tag{8}$$

- Differentiating (8) w.r.t. s leads to

$$\frac{\partial k^{\#}_{So}}{\partial s} = \frac{f(k^{\#}_{So})}{(\delta + \mu_{N^{\#}}) - s \cdot f'(k^{\#}_{So})} > 0, \tag{9}$$

since $(\delta + \mu_{N^{\#}}) > s \cdot f'(k^{\#})$ at the steady-state solution $k^{\#}_{So}$

- Thus, the **level** of $k^{\#}_{So}$ increases permanently. Similarly, the **level** of $y^{\#}_{So}$ increases permanently, ie

$$\frac{\partial y^{\#}_{So}}{\partial s} = f'(k^{\#}_{So}) \frac{\partial k^{\#}_{So}}{\partial s} > 0 \tag{10}$$

II Solow model: Theory

Effects of a change in the savings rate

Effect of a permanent change in the savings rate s on output:

- To verify the previous reasoning, consider the **Cobb-Douglas example:** Using $f(k^{\#}) = (k^{\#})^{\alpha}$, eqn (8) is given by

$$s \cdot (k^{\#}_{So})^{\alpha} = (\delta + \mu_{N^{\#}})k^{\#}_{So},$$

such that we can solve explicitly for $k^{\#}_{So}$, ie

$$k^{\#}_{So} = \left(\frac{s}{\delta + \mu_{N^{\#}}}\right)^{\frac{1}{1-\alpha}},$$

implying $\frac{\partial k^{\#}_{So}}{\partial s} > 0$, $\frac{\partial y^{\#}_{So}}{\partial s} > 0$

II Solow model: Theory

Effects of a change in the savings rate

Effect of a permanent change in the savings rate s on output:

- **On impact**, the increase in s implies that $\Delta k^{\#}_{t+1}$ will be positive, ie the **permanent increase in** s induces a **temporary increase in the growth rate of** $k^{\#}_t$ until the new and higher level of $k^{\#}_{So}$ has been reached

$\longrightarrow$ What does this imply for the behaviour of **per capita output** $y_t = \frac{Y_t}{N_t}$?

- As long as $k^{\#}_t$ increases, $\frac{Y_t}{N_t}$ grows both because A_t increases and because of the increase in $k^{\#}_t$ itself
- Eventually, however, the additional savings will be fully devoted to maintaining the higher level of $k^{\#}_{So}$ such that the growth rate of $\frac{Y_t}{N_t}$ will return to μ_A
- **In sum**, the permanent increase in s leads only to a temporary increase in the growth rate of $\frac{Y_t}{N_t}$, ie **per capita output benefits from a permanent level effect, but no permanent growth effect**

II Solow model: Theory

Effects of a change in the savings rate

Effect of a permanent change in the savings rate s on consumption:

$\longrightarrow$ The effect on consumption is less straightforward.
Consider first the effect on consumption in efficiency units (ie $c_t^{\#}$)

- **On impact**, the increase in s leads to a drop in $c_t^{\#}$ (since $k_t^{\#}$ is predetermined)
- The **long-run effect** on $c^{\#}$ is a priori ambiguous. Why?
  Recall from above

$$c_{So}^{\#} = f(k_{So}^{\#}) - (\delta + \mu_{N^{\#}}) \cdot k_{So}^{\#},$$

implying

$$\frac{\partial c_{So}^{\#}}{\partial s} = \underbrace{[f'(k_{So}^{\#}) - (\delta + \mu_{N^{\#}})]}_{\overset{?}{\gtrless}0} \cdot \underbrace{\frac{\partial k_{So}^{\#}}{\partial s}}_{>0} \qquad (11)$$

II Solow model: Theory

Effects of a change in the savings rate

Effect of a permanent change in the savings rate s on consumption:

Interpretation of the **ambiguity** in eqn (11), ie

$$\frac{\partial c^{\#}_{So}}{\partial s} = \underbrace{\left[f'(k^{\#}_{So}) - (\delta + \mu_{N^{\#}})\right]}_{\overset{?}{\gtrless}0} \cdot \underbrace{\frac{\partial k^{\#}_{So}}{\partial s}}_{>0}$$

- As discussed, the increase in s always raises $k^{\#}_{So}$
- However, the long-run effect on $c^{\#}_{So}$ will only be positive if the marginal product of capital exceeds the marginal increase in break-even investment $(\delta + \mu_{N^{\#}})$, ie if there is more than enough additional output to maintain $k^{\#}_{So}$ at its higher level such that consumption $c^{\#}_{So}$ can rise...
- ...but this does not have to be satisfied...

II Solow model: Theory

Effects of a change in the savings rate

Effect of a permanent change in the savings rate s on consumption:

- The **ambiguity of the long-run effect on consumption** (ie $\frac{\partial c^{\#}_{So}}{\partial s} \overset{?}{\gtrless} 0$) can be linked to the **golden-rule criterion**
- Recall that the golden-rule consumption level $c^{\#}_{GR}$ is the highest possible consumption level (*now:* in efficiency units) that is attainable, ie:

$$c^{\#}_{GR} = \max_{k^{\#}} [f(k^{\#}) - (\delta + \mu_{N^{\#}}) \cdot k^{\#}],$$

  implying that the golden-rule capital stock satisfies

$$f'(k^{\#}_{GR}) = \delta + \mu_{N^{\#}}$$

- In the Solow-model, because of the central assumption that the savings rate is exogenous, there is no reason to expect that $k^{\#}_{So}$ is equal to $k^{\#}_{GR}$. To the contrary, $k^{\#}_{So}$ may be larger or smaller than $k^{\#}_{GR}$, ie the sign of

$$\frac{\partial c^{\#}_{So}}{\partial s} = \underbrace{[f'(k^{\#}_{So}) - (\delta + \mu_{N^{\#}})]}_{\overset{?}{\gtrless} 0} \cdot \underbrace{\frac{\partial k^{\#}_{So}}{\partial s}}_{>0},$$

  is a priori ambiguous

II Solow model: Theory

Effects of a change in the savings rate

Effect of a permanent change in the savings rate s on consumption:

→ What does this reasoning imply for the behaviour of **per capita consumption** $c_t = \frac{C_t}{N_t}$?

→ Since we can express c_t as $c_t = A_t \cdot c_t^{\#}$ and since A_t is subject to exogenous growth, the implications of a change in s can be summarized as follows:

- **On impact**, the increase in s leads to a drop in c_t
- The **long-run effect** on c is a priori ambiguous, in the sense that per capita consumption experiences a **permanent level effect** which can be **positive** or **negative**
- However, per capita consumption receives **no permanent growth effect** (ie once the new steady-state has been reached, c_t will grow, again, at the rate μ_A)

III Solow model: Quantitative aspects and empirical work

→ Over decades, the Solow-model has been the starting point for a vast empirical literature on issues of economic growth

→ In particular, the literature has recognized that the Solow-model has **quantitative implications** for:

I) Intertemporal adjustment dynamics and convergence processes **within countries**

II) Comparisons of developments of per capita incomes and convergence processes **between countries**

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

→ As derived above, the Solow-model generates the **qualitative predictions** that a permanent **increase** in the **savings rate** leads to **permanent level effects**, but only **temporary growth effects**

→ To assess the plausibility of these predictions, they need to be **quantified**, ie:

- **1)** What is the likely **size of the level effect**, in particular, **in terms of output**?
- **2)** How long will it take until the adjustment to the new steady state will have been achieved? To put it differently: **What is the speed of convergence?**

→ Answers to these questions can be obtained from a (simple) **calibration** of the model at its steady state

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

1) Size of the long-run output effect of a permanent change in s:

- Recall from eqns (9) and (10) derived above, ie

$$\frac{\partial k_{So}^{\#}}{\partial s} = \frac{f(k_{So}^{\#})}{(\delta + \mu_{N^{\#}}) - s \cdot f'(k_{So}^{\#})} > 0$$

$$\frac{\partial y_{So}^{\#}}{\partial s} = f'(k_{So}^{\#}) \frac{\partial k_{So}^{\#}}{\partial s} > 0$$

- Combining both eqns we can express the long-run effect of a change in s on $y_{So}^{\#}$ as

$$\frac{\partial y_{So}^{\#}}{\partial s} = \frac{f'(k_{So}^{\#}) \cdot f(k_{So}^{\#})}{(\delta + \mu_{N^{\#}}) - s \cdot f'(k_{So}^{\#})} \tag{12}$$

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

1) Size of the long-run output effect of a permanent change in s:

- To better interpret and later onwards to calibrate eqn (12), ie

$$\frac{\partial y^{\#}_{So}}{\partial s} = \frac{f'(k^{\#}_{So}) \cdot f(k^{\#}_{So})}{(\delta + \mu_{N^{\#}}) - s \cdot f'(k^{\#}_{So})}$$

let us make two changes, ie in
step i) we will write it as an elasticity (by multiplying the eqn by $\frac{s}{y^{\#}_{So}}$),
step ii) we will substitute out for s by using the steady-state relationship (8), ie $s \cdot f(k^{\#}_{So}) = (\delta + \mu_{N^{\#}}) \cdot k^{\#}_{So}$:

$$\frac{\partial y^{\#}_{So}}{\partial s} \frac{s}{y^{\#}_{So}} = \frac{s}{f(k^{\#}_{So})} \cdot \frac{f'(k^{\#}_{So}) \cdot f(k^{\#}_{So})}{(\delta + \mu_{N^{\#}}) - s \cdot f'(k^{\#}_{So})}$$

$$\frac{\partial y^{\#}_{So}}{\partial s} \frac{s}{y^{\#}_{So}} = \frac{1}{f(k^{\#}_{So})} \cdot \frac{(\delta + \mu_{N^{\#}}) \cdot k^{\#}_{So} \cdot f'(k^{\#}_{So})}{(\delta + \mu_{N^{\#}}) - (\delta + \mu_{N^{\#}}) \cdot \frac{k^{\#}_{So} \cdot f'(k^{\#}_{So})}{f(k^{\#}_{So})}}$$

$$\frac{\partial y^{\#}_{So}}{\partial s} \frac{s}{y^{\#}_{So}} = \frac{k^{\#}_{So} \cdot f'(k^{\#}_{So})}{f(k^{\#}_{So})} \cdot \frac{1}{1 - \frac{k^{\#}_{So} \cdot f'(k^{\#}_{So})}{f(k^{\#}_{So})}} \tag{13}$$

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

1) Size of the long-run output effect of a permanent change in s:

- In eqn (13), ie

$$\frac{\partial y^{\#}_{So}}{\partial s}\frac{s}{y^{\#}_{So}} = \frac{k^{\#}_{So}\cdot f'(k^{\#}_{So})}{f(k^{\#}_{So})}\cdot\frac{1}{1-\frac{k^{\#}_{So}\cdot f'(k^{\#}_{So})}{f(k^{\#}_{So})}}$$

  on the LHS the term $\frac{\partial y^{\#}_{So}}{\partial s}\frac{s}{y^{\#}_{So}}$ denotes the **elasticity of output with respect to the savings rate**. On the RHS, the term $\frac{k^{\#}_{So}\cdot f'(k^{\#}_{So})}{f(k^{\#}_{So})}$ denotes the **elasticity of output with respect to capital.** Both elasticities are evaluated at the steady-state value $k^{\#}_{So}$

- For brevity, let $\alpha_K(k^{\#}_{So}) = \frac{k^{\#}_{So}\cdot f'(k^{\#}_{So})}{f(k^{\#}_{So})}$, implying

$$\frac{\partial y^{\#}_{So}}{\partial s}\frac{s}{y^{\#}_{So}} = \frac{\alpha_K(k^{\#}_{So})}{1-\alpha_K(k^{\#}_{So})} \tag{14}$$

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

1) Size of the long-run output effect of a permanent change in s:

$\rightarrow$ **Alternative interpretation** of $\alpha_K(k^{\#}_{So}) = \frac{k^{\#}_{So} \cdot f'(k^{\#}_{So})}{f(k^{\#}_{So})}$, used for calibration:

- **Assume** that **factor markets are competitive** and there are **no externalities,** ie capital will be paid its (social) marginal product
- Because of **constant returns to scale** (and using Euler's theorem):

$$F(k^{\#}_{So}, 1) = \underbrace{F_K(k^{\#}_{So}, 1)}_{r_{So}+\delta} \cdot k^{\#}_{So} + \underbrace{F_{N^{\#}}(k^{\#}_{So}, 1)}_{w^{N^{\#}}_{So}} \cdot 1.$$

Using $F(k^{\#}_{So}, 1) \equiv f(k^{\#}_{So})$ this is equivalent to

$$f(k^{\#}_{So}) = f'(k^{\#}_{So}) \cdot k^{\#}_{So} + w^{N^{\#}}_{So} \tag{15}$$

implying that

$$\alpha_K(k^{\#}_{So}) = \frac{k^{\#}_{So} \cdot f'(k^{\#}_{So})}{f(k^{\#}_{So})} \in (0, 1) \tag{16}$$

denotes the **share of income paid to capital along the balanced growth path**

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

Comment: Wages

- In eq (15) which captures how income is distributed to capital and labour along a balanced growth path, ie

$$f(k^{\#}_{So}) = f'(k^{\#}_{So}) \cdot k^{\#}_{So} + w^{N^{\#}}_{So}$$

  the constant term $w^{N^{\#}}_{So} = F_{N^{\#}}(k^{\#}_{So}, 1)$ denotes the **wage rate per unit of effective labour**

- Along a balanced growth path, the **wage rate per worker** (w) will grow over time, ie

$$w_t = A_t \cdot w^{N^{\#}}_{So}$$

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

Background: Derivation of factor shares from the aggregate prod. function

- The derivation of $\alpha_K(k^{\#}_{So}) = \frac{k^{\#}_{So} \cdot f'(k^{\#}_{So})}{f(k^{\#}_{So})}$ via eq (16) as the income share of capital along a balanced growth path can be equivalently derived from the aggregate production function:

$$Y_t = F(K_t, N^{\#}_t) = F_K(K_t, N^{\#}_t) \cdot K_t + F_{N^{\#}}(K_t, N^{\#}_t) \cdot N^{\#}_t$$

$$\Leftrightarrow 1 = \underbrace{\frac{F_K(K_t, N^{\#}_t) \cdot K_t}{Y_t}}_{\alpha_K} + \underbrace{\frac{F_{N^{\#}}(K_t, N^{\#}_t) \cdot N^{\#}_t}{Y_t}}_{\frac{F_N(K_t, N^{\#}_t) \cdot N_t}{Y_t} = \alpha_N}$$

- Along a balanced growth path: $K_t = N^{\#}_t \cdot k^{\#}_{So}$ and $Y_t = N^{\#}_t \cdot f(k^{\#}_{So})$
- Moreover, use that F_K and F_N are homogenous of degree zero, implying

$$1 = \underbrace{F_K(k^{\#}_{So}, 1) \frac{k^{\#}_{So}}{f(k^{\#}_{So})}}_{\alpha_K(k^{\#}_{So})} + \underbrace{\frac{F_{N^{\#}}(k^{\#}_{So}, 1)}{f(k^{\#}_{So})}}_{\alpha_N(k^{\#}_{So})}$$

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

1) Size of the long-run output effect of a permanent change in s:

- **Calibration of** $\alpha_K(k^{\#}_{So})$: in most advanced countries the income share of capital is about 1/3
- This value implies that the elasticity of output w.r.t. the savings rate

$$\frac{\partial y^{\#}_{So}}{\partial s}\frac{s}{y^{\#}_{So}} = \frac{\alpha_K(k^{\#}_{So})}{1-\alpha_K(k^{\#}_{So})}$$

  is about 1/2
- **What does this imply?** It implies, for example, that an increase in the savings rate by 10 percent (ie let's say from 10% of output to 11%) increases the long-run level of per capita output by about 5 percent (relative to the path without the change in s).
  Correspondingly, a significant increase in the savings rate by 50 percent (ie from 10% to 15%) raises long-run output only moderately by about 25 percent.

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

1) Size of the long-run output effect of a permanent change in s:

- **In sum: Quantitatively**, calibrations of the Solow-model imply that **significant changes in** s tend to have only **moderate effects** on the **level of output** along a **balanced growth path**
- There are two reasons for this:

$\rightarrow$ a 'small' value of $\alpha_K(k^{\#}_{So})$ implies that $s \cdot f(k^{\#}_{So})$ has a relatively strong curvature, ie any upward shift of the savings-curve moves the intersection with the break-even-investment line only by little
$\rightarrow$ a 'small' value of $\alpha_K(k^{\#}_{So})$ implies that the output effect of the change in $k^{\#}_{So}$ will be relatively small

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

2) Speed of convergence:

- The previous reasoning can be extended to quantify the predicted speed of convergence to the new steady state after a change in s, using a standard calibration
- **Idea:** Such measure can be established from a first-order Taylor expansion of the central eq (5), ie

$$k_{t+1}^{\#} \cdot (1 + \mu_{N^{\#}}) = s \cdot f(k_t^{\#}) + (1 - \delta) k_t^{\#},$$

around the (initial) steady state $k_{So}^{\#}$

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

2) Speed of convergence:

→ Consider eq (5), ie

$$k^{\#}_{t+1} \cdot (1 + \mu_{N^{\#}}) = s \cdot f(k^{\#}_t) + (1 - \delta) k^{\#}_t,$$

- Approximation of the LHS of eq (5):

$$k^{\#}_{So} \cdot (1 + \mu_{N^{\#}}) + (1 + \mu_{N^{\#}}) \cdot (k^{\#}_{t+1} - k^{\#}_{So})$$

- Approximation of the RHS of eq (5):

$$s \cdot f(k^{\#}_{So}) + (1 - \delta) k^{\#}_{So} + [s \cdot f'(k^{\#}_{So}) + (1 - \delta)] \cdot (k^{\#}_t - k^{\#}_{So})$$

- Combine these expressions to approximate eq (5) around $k^{\#}_{So}$ as:

$$k^{\#}_{t+1} - k^{\#}_{So} = \underbrace{\frac{s \cdot f'(k^{\#}_{So}) + (1 - \delta)}{1 + \mu_{N^{\#}}}}_{\lambda} \cdot (k^{\#}_t - k^{\#}_{So}) \qquad (17)$$

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

2) Speed of convergence:

- In eq (17), ie

$$k^{\#}_{t+1} - k^{\#}_{So} = \underbrace{\frac{s \cdot f'(k^{\#}_{So}) + (1-\delta)}{1+\mu_{N^{\#}}}}_{\lambda} \cdot (k^{\#}_{t} - k^{\#}_{So}),$$

  the eigenvalue λ governs the dynamics of the linearized first-order difference equation

- To establish the magnitude of λ let us use the known relationships

$$s = \frac{(\delta + \mu_{N^{\#}}) \cdot k^{\#}_{So}}{f(k^{\#}_{So})} \quad \text{and} \quad \alpha_K(k^{\#}_{So}) = \frac{k^{\#}_{So} \cdot f'(k^{\#}_{So})}{f(k^{\#}_{So})}$$

  to rewrite λ as

$$\lambda = \frac{(\delta + \mu_{N^{\#}}) \cdot \alpha_K(k^{\#}_{So}) + 1 - \delta}{1+\mu_{N^{\#}}} = \frac{1 + \alpha_K(k^{\#}_{So}) \cdot \mu_{N^{\#}} - \delta \cdot [1 - \alpha_K(k^{\#}_{So})]}{1+\mu_{N^{\#}}} \in (0,1) \tag{18}$$

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

2) Speed of convergence:

- As to be expected, eq (18) ensures $\lambda \in (0,1)$. This confirms that the linearized dynamics are stable, ie any deviation of $k_0^{\#}$ from the steady state value $k_{So}^{\#}$ will be scaled down over time and eventually disappears
- This can be seen if one writes eq (17) via repeated substitutions as

$$k_t^{\#} - k_{So}^{\#} = \lambda^t \cdot (k_0^{\#} - k_{So}^{\#}),$$

ie $\lambda \in (0,1)$ ensures that $k_t^{\#}$ converges against $k_{So}^{\#}$ since $\lim_{t \to \infty} \lambda^t = 0$

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

2) Speed of convergence:

Interpretation of λ :

- In eq (17), ie
$$k_t^{\#} - k_{So}^{\#} = \lambda^t \cdot (k_0^{\#} - k_{So}^{\#}),$$
λ is responsible for the convergence speed at which the economy moves to the steady state after a shock
- A high value of $\lambda \in (0, 1)$ implies a low convergence speed and vice versa
- **Half-life of the convergence process:**
Consider
$$\frac{k_t^{\#} - k_{So}^{\#}}{k_0^{\#} - k_{So}^{\#}} = \lambda^{t_{HL}} = 0.5 \qquad (19)$$
Then t_{HL} denotes the number of years after which half of the gap between the initial value $k_0^{\#}$ and the steady-state value $k_{So}^{\#}$ will have been closed, or, for short, the *half-life*

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

2) Speed of convergence:

- To **quantify** the likely **magnitude of** λ, go back to eq (18), ie

$$\lambda = \frac{1 + \alpha_K(k_{So}^{\#}) \cdot \mu_{N^\#} - \delta \cdot [1 - \alpha_K(k_{So}^{\#})]}{1 + \mu_{N^\#}},$$

which, for small values of $\mu_{N^\#}$, is approximately equal to

$$\begin{aligned} \lambda &\approx 1 + \alpha_K(k_{So}^{\#}) \cdot \mu_{N^\#} - \delta \cdot [1 - \alpha_K(k_{So}^{\#})] - \mu_{N^\#} \\ &\approx 1 - [1 - \alpha_K(k_{So}^{\#})] \cdot [\delta + \mu_{N^\#}] \end{aligned}$$

- **Standard calibration ranges** (long-run average annual values, industrialized countries like US or in Europe)**:**
Depreciation of capital (δ): $\approx 3 - 4\%$
Population growth (μ_N): $\approx 1 - 2\%$
Growth rate of per capita output (μ_A): $\approx 1 - 2\%$
Factor share of capital (α_K): $\approx 1/3$

$$\Longrightarrow \quad \lambda \approx 1 - (1 - \frac{1}{3}) \cdot 0.06 \approx 1 - 0.04 \approx 0.96$$

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

2) Speed of convergence:

$\rightarrow$ **Quantitative implications of the calibration** $\lambda \approx 0.96$:

- After **one year**, about 4% of the gap in terms of $k_0^{\#} - k_{So}^{\#}$ will be closed
- The half-life can be calculated from eq (19), ie

$$\frac{k_t^{\#} - k_{So}^{\#}}{k_0^{\#} - k_{So}^{\#}} = \lambda^{t_{HL}} = 0.5,$$

which implies

$$\begin{aligned} t_{HL} \cdot \ln(0.96) &= \ln(0.5) \\ t_{HL} &= \frac{\ln(0.5)}{\ln(0.96)} \approx 16.98 \end{aligned}$$

ie after about **17 years only 50%** of the **adjustment will have occurred**

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

2) Speed of convergence:

Comment: Output dynamics
The eigenvalue λ drives not only the linearized dynamics of the capital stock in terms of $k_t^\# - k_{So}^\#$, but also of output in terms of $y_t^\# - y_{So}^\#$

- Why? Consider $y_t^\# - y_{So}^\# = f(k_t^\#) - f(k_{So}^\#)$ which is approximately equal to

$$y_t^\# - y_{So}^\# \approx f'(k_{So}^\#) \cdot (k_t^\# - k_{So}^\#),$$

leading to

$$k_t^\# - k_{So}^\# \approx \frac{1}{f'(k_{So}^\#)} \cdot (y_t^\# - y_{So}^\#) \quad \text{and} \quad k_{t+1}^\# - k_{So}^\# \approx \frac{1}{f'(k_{So}^\#)} \cdot (y_{t+1}^\# - y_{So}^\#)$$

- Use these expressions in eq (17) to confirm

$$y_{t+1}^\# - y_{So}^\# = \lambda \cdot (y_t^\# - y_{So}^\#)$$

III Solow model: Quantitative aspects and empirical work

Implications for adjustment dynamics within countries

Summary assessment: quantitative implications for adjustment dynamics within countries

- For illustration, let us combine the **second prediction** of a **slow convergence speed** with the **first prediction** established above, namely that an increase in the savings rate by 10 percent (eg from 10% of output to 11%) leads to a rather modest **increase** in the **long-run level** of per capita **output** by about **5 percent** (relative to the path without the change in s)
- After **one year**, output will be about $0.04 \cdot 5\% \approx 0.2\%$ above the initial path
- After **17 years,** output will still only be about $0.5 \cdot 5\% \approx 2.5\%$ above the initial path
- In the (very) **long-run,** output will be about 5% above the initial path

→ **Quantitatively**, standard calibrations of the Solow-model predict that **changes in** s generate at best **modest output effects** which **occur very slowly**

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

Starting point for **comparisons between countries:**
$\rightarrow$ The Solow-model identifies two key **sources of variations in per capita output**

1) Differences in the **per capita levels of the capital stock** (ie $\frac{K}{N}$)
2) Differences in the **effectiveness of labour,** as captured by the level of A

- To develop quantitative predictions of the Solow-model from a cross-country perspective, let us initially explore the **assumption** that all differences in $\frac{Y}{N}$ between **otherwise identical countries** are entirely due to the first channel, ie **countries differ only in terms of** $\frac{K}{N}$
- This strong assumption generates clear-cut (and closely related) quantitative predictions for the paths of per capita output ($\frac{Y}{N}$):

Predictions:
(i) At any moment in time, observed **differences in the levels of** $\frac{Y}{N}$ are fully **explained by differences in** $\frac{K}{N}$
(ii) Over time, the levels of $\frac{Y}{N}$ display **absolute convergence** towards the same long-run levels

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

Prediction: (i) Differences in $\frac{Y}{N}$ to be explained by differences in $\frac{K}{N}$

$\rightarrow$ This prediction is not borne out by the data for standard calibrations of α

- **Illustration:** Cobb-Douglas production function

$$y_t^{\#} = (k_t^{\#})^{\alpha}$$

- Consider **two countries I (rich)** and **II (poor)** with output per worker levels

$$y_{I,t} = 10 \cdot y_{II,t},$$

ie output per worker in country I is 10 times larger than in country II

- To account for such differences: What is the **implied difference in capital per worker levels**, as captured by the factor X, ie

$$k_{I,t} = \underbrace{X}_{?} \cdot k_{II,t},$$

- Notice: $y_{I,t} = 10 \cdot y_{II,t}$ implies $y_{I,t}^{\#} = 10 \cdot y_{II,t}^{\#}$, since, by assumption, $A_{I,t} = A_{II,t}$

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

Prediction: (i) Differences in $\frac{Y}{N}$ to be explained by differences in $\frac{K}{N}$

- **Direct approach:**

$$y^{\#}_{I,t} = 10 \cdot y^{\#}_{II,t} = \underbrace{(X \cdot k^{\#}_{II,t})}_{k^{\#}_{I,t}}{}^{\alpha} = X^{\alpha} \cdot \underbrace{(k^{\#}_{II,t})^{\alpha}}_{y^{\#}_{II,t}},$$

  ie

$$10 = X^{\alpha}$$

- A capital share of $\alpha = 1/3$ implies a factor of $X = 1000$, ie capital per worker in the rich country needs to be 1000 times larger than in the poor country to account for a tenfold difference in output per worker
- A higher capital share of $\alpha = 1/2$ (which is above what any income data would suggest) still leads to a factor of $X = 100$

$\rightarrow$ **Summary assessment:** there is no evidence for such strong differences in $\frac{K}{N}$ between countries, ie the observed differences in capital per worker are far smaller than those that would be needed to account for the differences in output per worker that we seek to understand

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

Prediction: (i) Differences in $\frac{Y}{N}$ to be explained by differences in $\frac{K}{N}$

- There exists also an **indirect approach** to support this assessment, in the sense that the variation in $\frac{K}{N}$ that would be needed to account for observed differences in output per capita implies **implausibly large differences in the rate of return on capital** (see Lucas, 1990)
- **Idea:** start out from $y_t^{\#} = (k_t^{\#})^{\alpha}$ and consider

$$\begin{aligned} f'(k_t^{\#}) &= \alpha \cdot (k_t^{\#})^{\alpha-1} \\ &= \alpha \cdot (y_t^{\#})^{\frac{\alpha-1}{\alpha}} \end{aligned}$$

- Assume, again,

$$y_{I,t} = 10 \cdot y_{II,t},$$

  ie output per worker in country I is 10 times larger than in country II
- To account for such differences: What is the **implied difference in the marginal product of capital,** as captured by the factor $\widetilde{X}$, ie

$$f'(k_{I,t}^{\#}) = \underbrace{\widetilde{X}}_{?} \cdot f'(k_{II,t}^{\#}),$$

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

Prediction: (i) Differences in $\frac{Y}{N}$ to be explained by differences in $\frac{K}{N}$

- **Indirect approach:**

$$f'(k^{\#}_{I,t}) = \alpha \cdot (y^{\#}_{I,t})^{\frac{\alpha-1}{\alpha}} = \alpha \cdot (10 \cdot y^{\#}_{II,t})^{\frac{\alpha-1}{\alpha}} = 10^{\frac{\alpha-1}{\alpha}} \cdot \underbrace{\alpha \cdot (y^{\#}_{II,t})^{\frac{\alpha-1}{\alpha}}}_{f'(k^{\#}_{II,t})},$$

ie

$$\widetilde{X} = 10^{\frac{\alpha-1}{\alpha}} = \left(\frac{1}{10}\right)^{\frac{1-\alpha}{\alpha}}$$

- A capital share of $\alpha = 1/3$ implies a factor of $\widetilde{X} = 1/100$, ie the marginal product of capital in the rich country needs to be 100 times smaller than in the poor country to account for a tenfold difference in output per worker
- A higher capital share of $\alpha = 1/2$ (which is above what any income data would suggest) still leads to a factor of $\widetilde{X} = 1/10$
- Notice that the real interest rate is closely related to $f'(k^{\#}_t)$ via the expression $r_t = f'(k^{\#}_t) - \delta$

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

Prediction: (i) Differences in $\frac{Y}{N}$ to be explained by differences in $\frac{K}{N}$

Summary assessment of the indirect approach:
($\rightarrow$ see: Lucas, R., Why doesn't capital flow from rich to poor countries?, *American Economic Review*, 80, 92-96, 1990)

- Lucas (1990) concludes that there is no evidence for such strong differences in return rates between countries: if return rates in poor countries exceeded those in rich countries by a factor 100 (or even 10) there would be significant incentives to invest in poor countries.
- Even if one recognizes the existence of capital-market imperfections, fears of unsecure ownership rights or even expropriations, such differences should support massive capital flows from the rich to the poor countries. But we don't see such strong flows!
- In other words, the observed differences in return rates between countries are far smaller than those that would be needed to account for the differences in output per worker that we seek to understand

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

Prediction: (ii) Absolute convergence

- From the perspective of the Solow-model, the **assumption** that differences in per capita income levels (ie $\frac{Y}{N}$) between **otherwise identical countries** are entirely driven by differences in $\frac{K}{N}$ has strong implications for the convergence of per capita incomes
- According to this assumption, countries should be seen to start out from different historical starting positions relative to the same balanced growth path
- Over time countries should converge to this balanced growth path, ie
  $\longrightarrow$ during the catching-up period **poor countries should have higher per capita growth rates than rich countries**
  $\longrightarrow$ in the long run, all countries should display **absolute convergence** of per capita incomes towards the same long-run levels

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

Prediction: (ii) Absolute convergence

- To assess the evidence concerning the hypothesis of absolute convergence it is instructive to look at findings from studies by **Baumol (1986)** and **DeLong (1988)** and to draw on *Figures 1.7-1.9* in the textbook by Romer (2006), p.33 f.
- The influential study by Baumol (1986) shows that the hypothesis of absolute convergence is a legitimate hypothesis, ie at least for some data it does have a certain support
- The follow-up study by DeLong (1988), however, reveals a number of shortcomings of the analysis used by Baumol and concludes that there is no support for the hypothesis

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

Prediction: (ii) Absolute convergence

Results from Baumol (1986):

- Baumol examines convergence from 1870 to 1979 among 16 industrialized countries (see *Figure 1.7*). To account for growth rates of output per capita he estimates

$$\ln[(\frac{Y}{N})_{i,1979}] - \ln[(\frac{Y}{N})_{i,1870}] = a + b \cdot \ln[(\frac{Y}{N})_{i,1870}] + \varepsilon_i, \qquad (20)$$

  ie he regresses the growth rate of $\frac{Y}{N}$ (of countries with index i) between 1870 and 1979 on a constant and the initial income level in the year 1870
- The parameter b is the crucial one for the convergence debate, ie if
  $b = 0$: **no convergence** (since growth is uncorrelated with initial income)
  $b = -1$: **absolute convergence** (since higher initial income lowers growth one-to-one such that output per capita in 1979 is uncorrelated with the starting value in 1870)
  $b \in (0, -1)$: indicates some **convergence** since countries with higher initial incomes have lower growth
- Main finding of Baumol: The **estimate of** b is highly significant and **very close to** -1**, supporting the hypothesis of absolute convergence**

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

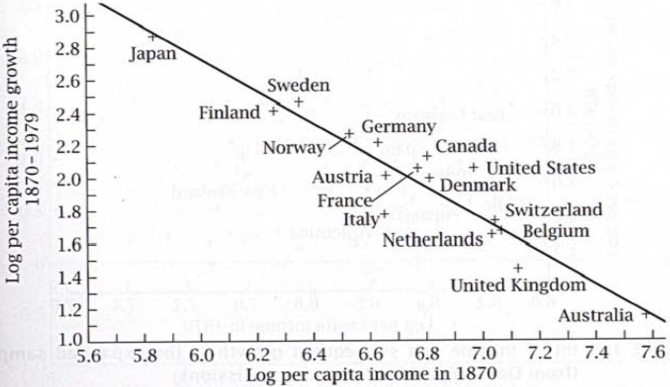

FIGURE 1.7 Initial income and subsequent growth in Baumol's sample (from DeLong, 1988; used with permission)

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

Prediction: (ii) Absolute convergence

Results from DeLong (1988):

- **Claim:** Baumol's main finding is misleading, since it suffers from 2 problems: i) **sample selection** and ii) **measurement error**
- **Sample selection:** historical data often constructed retrospectively, ie
  $\longrightarrow$ countries that were poor around 1870 are likely to be included only if they grew rapidly afterwards
  $\longrightarrow$ countries that were rich around 1870 are likely to be included, even if they grew relatively slowly afterwards
- If this sample selection conjecture is correct, this leads to a bias in favour of the hypothesis of absolute convergence although on average such support should not be given

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

Prediction: (ii) Absolute convergence

Results from DeLong (1988):

- How to eliminate the sample selection bias?
  → Lack of data makes it impossible to include all countries
  → Yet, adding a number of additional countries which were around 1870 at least as rich as the relatively poor ones in Baumol's sample goes a long way to remove the bias
  → in particular, adding countries like Argentina, Chile, Ireland, New Zealand, Portugal, Spain (see *Figure 1.8*) changes significantly the estimate of b which drops to -0.57
- **Upshot:** correcting for the sample selection bias weakens significantly Baumol's initial finding in support of absolute convergence

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

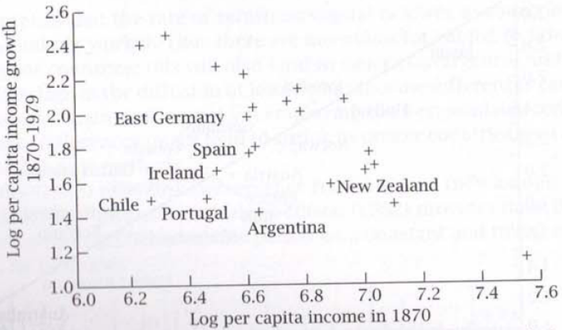

FIGURE 1.8 Initial income and subsequent growth in the expanded sample (from DeLong, 1988; used with permission)

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

Prediction: (ii) Absolute convergence

Results from DeLong (1988):

- A second concern relates to **measurement error,** since income data for 1870 are surely measured with a high degree of imprecision
- Measurement error introduces a (second) bias in favour of the hypothesis of absolute convergence, ie
  whenever 1870 income is overstated subsequent growth between 1870-1979 will be understated and
  whenever 1870 income is understated subsequent growth between 1870-1979 will be overstated
- In other words, strongly negative estimates of b may well stand for two very different things:
  $\longrightarrow$ if the measurement error is insignificant there is strong true convergence
  $\longrightarrow$ if the measurement error is significant there is weak or no true convergence

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

Prediction: (ii) Absolute convergence

Results from DeLong (1988):

- When re-estimating eq (20), DeLong (1988) allows for various degrees of measurement error
- Finding: even a moderate degree of measurement error leads to strong changes in the estimate of b (such that b drops further and approaches 0)
- In other words: when allowing for both sample selection bias and measurement error, this eliminates Baumol's estimate in favour of absolute convergence

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

Prediction: (ii) Absolute convergence

Summary assessment:

- For large samples of diverse countries and over different periods of time the strong prediction of **absolute convergence** of per capita incomes is **not supported by the data**
- In this spirit, see the scatterplot in *Figure 1.9* which covers virtually all countries (except from the Communist world) for the period 1960-2000

III Solow model: Quantitative aspects and empirical work

Implications for comparisons between countries

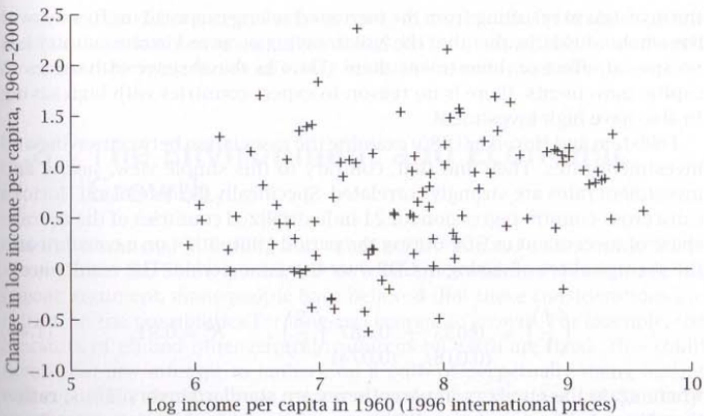

FIGURE 1.9 Initial income and subsequent growth in the postwar period