Lecture 3
Solving linearized systems of difference equations with backwardlooking and forwardlooking variables (Technical Session I)

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I Motivation

- As derived in Lecture 2, the **dynamics of the centralized economy** are governed by the pair of equations

$$U'(c_t) = \beta U'(c_{t+1})[f'(k_{t+1}) + (1-\delta)] \quad (1)$$
$$c_t = f(k_t) - k_{t+1} + (1-\delta)k_t, \quad (2)$$

The system (1)-(2) constitutes a non-linear two-dimensional dynamic system in c and k with one initial condition (k_0) and one terminal condition, as given by the transversality condition

$$\lim_{t\to\infty} \beta^t \cdot U'(c_t) \cdot k_{t+1} = 0$$

$\rightarrow$ k is the single (backward-looking) state variable (with predetermined initial value k_0)
$\rightarrow$ c is the single (forward-looking) control variable w/o initial condition

- Using the **graphical representation of a phase diagram**, Lecture 2 had illustrated that the linearized dynamics of (1) and (2) are stable in a particular sense, ie the system is **saddlepath-stable** around the unique steady state characterized by

$$f'(k^*) = \delta + \theta \quad \text{and} \quad c^* = f(k^*) - \delta k^*$$

I Motivation

Goal of this Lecture:

$\rightarrow$ i) Derive **analytically** the saddlepath-stable solution of the **linearized dynamics** around (k^*, c^*)

$\rightarrow$ ii) Extend the reasoning to a **general classification of stability patterns of higher-dimensional linearized systems** where we have n_1 predetermined and $n_2 = n - n_1$ forwardlooking variables

$\rightarrow$ iii) Comment on the **economics behind saddlepath-stable dynamics**

II Centralized economy: Solution of the linearized dynamics

- The dynamic system (1)-(2) is non-linear. 'Way out'?
- Analysis of a linearized system, obtained from a 1st-order Taylor expansion of (1)-(2) around the unique steady state (k^*, c^*) :

$$\begin{bmatrix} c_{t+1} - c^* \\ k_{t+1} - k^* \end{bmatrix} = A \cdot \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix} \tag{3}$$

- A is a 2x2-matrix, with coefficients evaluated at the steady state, ie

$$A = \begin{bmatrix} a_{11}(k^*, c^*) & a_{12}(k^*, c^*) \\ a_{21}(k^*, c^*) & a_{22}(k^*, c^*) \end{bmatrix}$$

II Centralized economy: Solution of the linearized dynamics

For the particular eqns (1) and (2), the matrix A in eqn (3) can be written as

$$\begin{bmatrix} c_{t+1} - c^* \\ k_{t+1} - k^* \end{bmatrix} = A \cdot \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix} \quad \text{with:}$$

$$A = \begin{bmatrix} 1 + \beta \cdot f''(k^*) \cdot \frac{U'(c^*)}{U''(c^*)} & -f''(k^*) \cdot \frac{U'(c^*)}{U''(c^*)} \\ -1 & f'(k^*) + 1 - \delta \end{bmatrix} \tag{4}$$

II Centralized economy: Solution of the linearized dynamics

Analytical characterization of the (in)stability of linearized systems:

$\longrightarrow$ In general, the (in)stability of linearized systems of difference equations is determined by their characteristic roots or, equivalently, their eigenvalues, denoted by λ

$\longrightarrow$ A *nxn*-system has generically *n* distinct eigenvalues (and, for simplicity, we consider $|\lambda_i| \neq 1$)

II Centralized economy: Solution of the linearized dynamics

Analytical characterization of the (in)stability of linearized systems: special case of a single first-order difference equation

Consider some variable x_t which evolves according to:

$$x_{t+1} - x^* = \lambda \cdot (x_t - x^*) \tag{5}$$

$\longrightarrow$ The eigenvalue λ induces a linear mapping such that the scalar argument $(x_t - x^*)$ is scaled up or down over time, depending on whether $|\lambda| \gtrless 1$

Backwardlooking interpretation:

Assume $|\lambda| < 1$: stability for arbitrary initial condition $x_t \neq x^*$

Forwardlooking interpretation:

Assume $|\lambda| > 1$:

Moreover, assume x is specified as a forwardlooking variable w/o initial, but with terminal condition $\lim_{T\to\infty} (x_{t+T} - x^*) = 0$

$\longrightarrow$ Stability for the particular initial value $x_t = x^*$. Why? Rewrite eqn (5) as

$$x_t - x^* = \frac{1}{\lambda}(x_{t+1} - x^*) = (\frac{1}{\lambda})^T \cdot (x_{t+T} - x^*),$$

implying $x_t = x^*$ since the term $x_{t+T} - x^*$ is bounded by the terminal condition $\lim_{T\to\infty} (x_{t+T} - x^*) = 0$

II Centralized economy: Solution of the linearized dynamics

Analytical characterization of the (in)stability of linearized systems:

Consider now:

$$\begin{bmatrix} c_{t+1} - c^* \\ k_{t+1} - k^* \end{bmatrix} = A \cdot \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix}$$

$\rightarrow$ Is there a counterpart to the just discussed case of a first-order difference equation (with a single eigenvalue λ) for the 2x2-system governed by A?

$\rightarrow$ To simplify notation let $h_{t+1} = A \cdot h_t$ with: $h_t \equiv \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix}$

$\rightarrow$ **Special case:** Assume

$$A \cdot h_t = \lambda \cdot h_t = h_{t+1},$$

ie the matrix A induces a linear mapping such that the vector argument h_t is scaled up or down over time, depending on whether $|\lambda| \gtrless 1$

In such special case denotes:

i) the scalar λ an **eigenvalue** of the matrix A

ii) the vector $h \equiv q$ an **eigenvector** of A, associated with the eigenvalue λ

II Centralized economy: Solution of the linearized dynamics

Analytical characterization of the (in)stability of linearized systems:

$\longrightarrow$ From the eqn

$$A \cdot q = \lambda \cdot q$$

eigenvalues solve the equation

$$[A - \lambda I] \cdot q = 0, \quad \text{with: } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\longrightarrow$ For non-trivial solutions (ie $q \neq 0$), the matrix $[A - \lambda I]$ needs to be 'singular' (ie the inverse of $[A - \lambda I]$ does not exist), leading to the so-called **characteristic equation**:

$$|A - \lambda I| = 0 \quad \Leftrightarrow \quad \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

Equivalently, the characteristic equation can be written as

$$\lambda^2 - \underbrace{(a_{11} + a_{22})}_{Tr(A)}\lambda + \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{Det(A)} = 0 \tag{6}$$

II Centralized economy: Solution of the linearized dynamics

Analytical characterization of the (in)stability of linearized systems:

→ The characteristic eqn (6) is a quadratic eqn in λ
→ There exist generically two different eigenvalues λ_1 and λ_2, ie

$$\lambda_{1,2} = \frac{1}{2} \cdot Tr(A) \pm \frac{1}{2} \cdot \sqrt{(Tr(A))^2 - 4 \cdot Det(A)}$$

→ with associated eigenvectors $q_1 = \begin{pmatrix} \mu_1 \\ \overline{q}_1 \cdot \mu_1 \end{pmatrix}$ and $q_2 = \begin{pmatrix} \mu_2 \\ \overline{q}_2 \cdot \mu_2 \end{pmatrix}$
→ since each λ_i generates 2 linearly dependent equations, the associated eigenvectors have a unique direction (via $\overline{q}_i$), but not a particular length

Some simplifying **notation**:
→ 2x2−**Matrix** Q **of stacked eigenvectors:**

$$Q = [q_1 \ q_2] = \begin{bmatrix} \mu_1 & \mu_2 \\ \overline{q}_1 \cdot \mu_1 & \overline{q}_2 \cdot \mu_2 \end{bmatrix}$$

→ 2x2−**Diagonal matrix** Λ **of eigenvalues:**

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

II Centralized economy: Solution of the linearized dynamics

Analytical characterization of the (in)stability of linearized systems:

$\longrightarrow$ Write the definition of eigenvalues and eigenvectors in matrix form:

$$A \cdot Q = A \cdot [q_1 \ q_2] = [q_1 \ q_2] \cdot \left[\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array} \right] = Q \cdot \Lambda$$

$\longrightarrow$ Since $Q \cdot Q^{-1} = I$, rewrite the matrix A via its **'Jordan canonical form'**:

$$A = Q \cdot \Lambda \cdot Q^{-1},$$

where it is customary to order the eigenvalues in Λ by size (starting with the smallest one in the top left corner of Λ)

$\longrightarrow$ The **inverse matrix** Q^{-1} of Q is also 2x2-matrix**:**

$$Q^{-1} = \frac{1}{Det(Q)} \left[\begin{array}{cc} \overline{q}_2 \cdot \mu_2 & -\mu_2 \\ -\overline{q}_1 \cdot \mu_1 & \mu_1 \end{array} \right] \equiv \left[\begin{array}{cc} \widetilde{q_{11}} & \widetilde{q_{12}} \\ \widetilde{q_{21}} & \widetilde{q_{22}} \end{array} \right]$$

II Centralized economy: Solution of the linearized dynamics

Analytical characterization of the (in)stability of linearized systems:

$\longrightarrow$ Define a **new vector** z_t containing linear combinations of the initial variables with weights taken from Q^{-1} such that

$$z_t = \begin{pmatrix} z_{1,t} \\ z_{2,t} \end{pmatrix} = Q^{-1} \cdot h_t \,,$$

ie

$$z_{1,t} = \widetilde{q_{11}} \cdot h_{1,t} + \widetilde{q_{12}} \cdot h_{2,t} \text{ and } z_{2,t} = \widetilde{q_{21}} \cdot h_{1,t} + \widetilde{q_{22}} \cdot h_{2,t}$$

$\longrightarrow$ Rewrite the initial $2x2-$system (3), ie

$$h_{t+1} = A \cdot h_t \,,$$

using $A = Q \cdot \Lambda \cdot Q^{-1}$ as

$$Q^{-1} \cdot h_{t+1} = z_{t+1} = \Lambda \cdot z_t \tag{7}$$

Notice: Since Λ is a diagonal matrix, eqn (7) consists of two 'de-coupled' first-order difference eqns, qualitatively similar to (5), ie we can write it as

$$\begin{aligned} z_{1,t+1} &= \lambda_1 \cdot z_{1,t} \\ z_{2,t+1} &= \lambda_2 \cdot z_{2,t} \end{aligned}$$

II Centralized economy: Solution of the linearized dynamics

Analytical characterization of the (in)stability of linearized systems:

$\longrightarrow$ The pair of equations

$$z_{1,t+1} = \lambda_1 \cdot z_{1,t} \quad \text{and} \quad z_{2,t+1} = \lambda_2 \cdot z_{2,t} \tag{8}$$

describe the **general solution** of the $2x2-$system

$$h_{t+1} = A \cdot h_t$$

$\longrightarrow$ **Equivalently**, the general solution can be written as

$$h_t = \begin{pmatrix} h_{1,t} \\ h_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \overline{q}_1 \cdot \mu_1 \end{pmatrix} \cdot \lambda_1^t + \begin{pmatrix} \mu_2 \\ \overline{q}_2 \cdot \mu_2 \end{pmatrix} \cdot \lambda_2^t \tag{9}$$

$\longrightarrow$ Using either (8) or (9), the **definite solution** can be obtained if one uses the initial and terminal conditions

II Centralized economy: Solution of the linearized dynamics

Analytical characterization of the (in)stability of linearized systems:
$\rightarrow$ **Recall:** one predetermined variable (k) and one forwardlooking variable (c)
$\rightarrow$ **Assume:** $|\lambda_1| < 1$ and $|\lambda_2| > 1$

[In class it will be verified that the matrix A stated in eqn (4) generically satisfies this pattern of eigenvalues]

Since $|\lambda_2| > 1$ solve the second eqn $\quad z_{2,t+1} = \lambda_2 \cdot z_{2,t}$
forward, ie rewrite it as

$$z_{2,t} = \frac{1}{\lambda_2} \cdot z_{2,t+1} = \left(\frac{1}{\lambda_2}\right)^T \cdot z_{2,t+T}$$

and deduce from $\lim\limits_{T\to\infty} \left(\frac{1}{\lambda_2}\right)^T \cdot z_{2,t+T} = 0$ the solution

$$z_{2,t} = \widetilde{q_{21}} \cdot \underbrace{h_{1,t}}_{c_t - c^*} + \widetilde{q_{22}} \cdot \underbrace{h_{2,t}}_{k_t - k^*} = 0,$$

implying that the **forwardlooking** (control) **variable** c should be set s.t.

$$c_t - c^* = -\frac{\widetilde{q_{22}}}{\widetilde{q_{21}}} \cdot (k_t - k^*) \tag{10}$$

II Centralized economy: Solution of the linearized dynamics

Analytical characterization of the (in)stability of linearized systems:

$\longrightarrow$ What about the dynamics in $(k_t - k^*)$?
$\longrightarrow$ Use the first eqn

$$z_{1,t+1} = \lambda_1 \cdot z_{1,t} \quad \text{with:} \quad z_{1,t} = \widetilde{q_{11}} \cdot h_{1,t} + \widetilde{q_{12}} \cdot h_{2,t}$$

$\longrightarrow$ Substitute eqn (10),ie

$$\underbrace{c_t - c^*}_{h_{1,t}} = -\frac{\widetilde{q_{22}}}{\widetilde{q_{21}}} \cdot \underbrace{(k_t - k^*)}_{h_{2,t}}.$$

in the first eqn to obtain

$$[\widetilde{q_{12}} - \widetilde{q_{11}} \frac{\widetilde{q_{22}}}{\widetilde{q_{21}}}] \cdot (k_{t+1} - k^*) = \lambda_1 \cdot [\widetilde{q_{12}} - \widetilde{q_{11}} \frac{\widetilde{q_{22}}}{\widetilde{q_{21}}}] \cdot (k_t - k^*),$$

implying for the **law of motion** of the **state variable** k :

$$k_{t+1} - k^* = \lambda_1 \cdot (k_t - k^*) \tag{11}$$

II Centralized economy: Solution of the linearized dynamics

Saddlepath-stable solution:

$\longrightarrow$ The two eqns (10) and (11), ie

$$\begin{aligned} k_{t+1} - k^* &= \lambda_1 \cdot (k_t - k^*) \\ c_t - c^* &= -\frac{\widetilde{q_{22}}}{\widetilde{q_{21}}} \cdot (k_t - k^*) \end{aligned}$$

are the solutions, summarizing $\forall t \geqslant 0$ the behaviour of the linearized versions of (1) and (2), as captured by the matrix A along the linear saddlepath until convergence of k_t and c_t against k^* and c^*

$\longrightarrow$ The derivation of (10) and (11) has used that we have 1 stable and 1 unstable eigenvalue which we have matched with the single initial and the single terminal condition

II Centralized economy: Solution of the linearized dynamics

Initializing the system at $t = 0$:

$\longrightarrow$ Recall: k_0 is the single initial condition of the system (10) and (11)
$\longrightarrow$ Consider the two eqns at $t = 0$, ie

$$\begin{aligned} k_1 - k^* &= \lambda_1 \cdot (k_0 - k^*) \\ c_0 - c^* &= -\frac{\widetilde{q_{22}}}{\widetilde{q_{21}}} \cdot (k_0 - k^*), \end{aligned}$$

implying that we managed to initialize the law of motion for k_t and c_t by the single initial condition k_0
$\longrightarrow$ for all $t > 0$: unique values of k_t and c_t determined recursively by (10) and (11)

III Comments and Generalizations

Comments

Interpretation:

- The analyzed model of the centralized economy (which is a benchmark model of modern intertemporal macroeconomics) exhibits a unique steady state with (locally) saddlepath stable dynamics, ie by combining the restrictions from both initial and terminal conditions the dynamics of all variables are stable and uniquely defined around this steady state.
- This concept is a standard one which is routinely used in macro-models with forward-looking agents
- In **large-scale macro models** (used for forecasts and policy simulations), such configuration cannot be confirmed in simple phase diagrams. Instead, these models need to be solved numerically. Yet, the basic intuition for the possibility of saddlepath–stable dynamics of such systems is qualitatively in line with the above given analysis
- In **stochastic extensions** of models of this type it implies that small shocks (within the neighbourhood around a steady state) trigger stable and predictable reactions of optimizing agents such that the economy eventually returns to the starting point

III Comments and Generalizations

Comments

Cross-equation restriction and **Lucas critique**:

- Equations of type (10), ie

$$c_t - c^* = -\frac{\widetilde{q_{22}}}{\widetilde{q_{21}}} \cdot (k_t - k^*)$$

  are examples of **cross equation restrictions**
- Restrictions of this type, going back to Lucas (1976), are a key feature of macro-models which incorporate forwardlooking behaviour (*here:* captured by c) and are intimately linked to the so-called **Lucas critique**
- This critique revolutionized macroeconomic analysis 40 years ago
- In general, the Lucas critique says that econometricians who want to estimate a relationship like (10) need to be aware that coefficients like $-\widetilde{q_{22}}/\widetilde{q_{21}}$ consist not only of **structural ('deep') parameters** like β or δ, but typically also of **policy parameters** (like tax rates or parameters characterizing monetary policy)
- In particular, changes in parameters of policy rules do affect such coefficients, implying that **policy advice based on past estimates of such coefficients will be systematically wrong**

III Comments and Generalizations

Comments

Cross-equation restriction and **Lucas critique**:

- **Remark:** for the **particular version of the centralized economy** studied so far the Lucas critique does not apply since we have not yet introduced any policy decision (eg by fiscal or monetary policymakers), ie for this very special system the dynamics governed by A do not depend on any policy parameter
- However, already in slightly **extended model versions** which allow for policy choices the Lucas critique does apply. In other words, the coefficient linking consumption and capital (and, hence, output) will typically be a function not only of structural parameters like β or δ but also of policy parameters like tax rates or money growth rates etc.
- **Lucas critique:** In case policymakers announce a systematic change in their policy rule, forwardlooking agents will incorporate this in their decisions. Policy-advice not internalizing this reaction will be systematically misleading.

III Comments and Generalizations

Comments

Extensions, criticism and alternative views:

- For saddlepath-stable configurations, the role of the ‘fundamentals of the economy’ (here captured by the single value k_0) is very strong (and for many applications too strong)
- The **linearization** of macroeconomic models, while often inevitable, can come at a significant cost since the **‘global’** behaviour of economies can be very different from predictions obtained from **‘local’** characterizations.
- In particular, for many applications it may well happen that there exist **multiple steady states,** leading to global coordination problems and questions of equilibrium selection. Such issues are at odds with the strong uniqueness property of saddlepath-stable solutions

III Comments and Generalizations

Comments

Extensions, criticism and alternative views:

- **Alternative view:**
  $\longrightarrow$ Models should allow for **self-fulfilling fluctuations**, driven by non-fundamental 'animal spirits' (Keynes).
  $\longrightarrow$ With equally simple model ingredients, this can be achieved if the dynamics implied by the system of difference equations are somewhat different, leading to locally **indeterminate (but still stable) dynamics**
- More far-reaching criticism:
  $\longrightarrow$ **rational expectations assumption as such to be modified** (eg via learning) or entirely abandoned

III Comments and Generalizations

Generalizations

Generalization I (Large-scale deterministic linear systems):

$\longrightarrow$ Consider an economy characterized by n_1 **predetermined (or state) variables with initial conditions** and $n_2 = n - n_1$ **forwardlooking (or control) variables with terminal conditions** s.t.

$$h_{t+1} = \begin{bmatrix} h_{t+1}^P \\ h_{t+1}^F \end{bmatrix} = A \cdot \begin{bmatrix} h_t^P \\ h_t^F \end{bmatrix} = A \cdot h_t, \tag{12}$$

where A is a $nxn-$matrix, h is a $nx1-$vector and h^P and h^F are n_1x1 and n_2x1-vectors of predetermined and forwardlooking variables, respectively

III Comments and Generalizations

Generalizations

Generalization I (Large-scale deterministic linear systems):

Blanchard-Kahn (1980) conditions:

- If the system is to have a **unique stationary equilibrium**, n_1 eigenvalues of the matrix A need to satisfy $|\lambda_i| < 1$, $i = 1, 2, .., n_1$, while n_2 eigenvalues need to satisfy $\left|\lambda_j\right| > 1$, $j = n_1 + 1, .., n$.
- If there are fewer than n_2 eigenvalues with $\left|\lambda_j\right| > 1$, then the system is characterized by **multiple stationary equilibria (indeterminacy)**
- If there are more than n_2 eigenvalues with $\left|\lambda_j\right| > 1$, then **no solution exists**

- If a **unique stationary equilibrium** exists, the solution takes the form:

$$h_{t+1}^P = M \cdot h_t^P \quad \text{and} \quad h_t^F = C \cdot h_t^P$$

- If there exist **multiple stationary equilibria (indeterminacy)**:
  $\rightarrow$ possibility of **self-fulfilling fluctuations ('animal spirits')**

III Comments and Generalizations

Generalizations

Comment 1: Unit roots

- If eigenvalues satisfy the borderline case of $|\lambda_i| = 1$ ('unit root'), the classification can be adjusted:
  If the system is to have a **unique equilibrium**, n_1 eigenvalues of the matrix A need to satisfy $|\lambda_i| \leq 1$, $i = 1, 2, .., n_1$, while n_2 eigenvalues need to satisfy $\left|\lambda_j\right| > 1$, $j = n_1 + 1, .., n$.
- **Intuition:** Eigenvalues satisfying $|\lambda_i| = 1$ create special dynamics in the sense that the system will not return to its starting point, but neither will it explode
- **Numerically**, such constellation is not generic (ie the probability that we hit such special value for 'arbitrary' matrices A is zero)
- However, many models have deliberately a **theoretical** design such that unit roots do matter (eg permanent as opposed to transitory technology or taste shocks etc)

III Comments and Generalizations

Generalizations

Comment 2: Level changes vs. percentage deviations

- Typically, to make reactions between the various variables comparable, the representative entries of h_t^P and h_t^F are specified as **percentage deviation** of some variable from its steady state, like, eg,

$$h_i^P = \widehat{k}_t = \frac{k_t - k^*}{k^*} \quad \text{or} \quad h_j^F = \widehat{c}_t = \frac{c_t - c^*}{c^*},$$

  and not the absolute differences (as done above)

- Variables with a **hat-notation** ($\widehat{k}_t$, $\widehat{c}_t$ etc.) typically describe such percentage deviation
- This change in representation matters only at the stage when the linearizations are done, but not afterwards