Lecture 2
The Centralized Economy: Basic features

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I Motivation

- This Lecture introduces the basic dynamic general equilibrium model of a closed economy which is at the heart of modern macroeconomics

→ Main reference: Wickens, Chapter 2, Sections 2.1-2.4

- Goal: we will analyze how to optimally allocate output between consumption and investment (ie capital accumulation) or, alternatively, between ‘consumption today’ and ‘consumption tomorrow’
I Motivation

In this lecture we will isolate a few core aspects. Many important things will be missing. For example:

- there will be no government, no market structure (in particular: no financial markets), no money (such that all variables are in real, not in nominal terms)
- there will be no uncertainty and no sources of persistence
- the labour supply will be fixed and capital can be installed without adjustment costs
- there will be no population growth and no technical progress
I Motivation

Why do we start with such a seemingly unrealistic and simplistic macroeconomic model?

- There is a good scientific tradition to start out from simple, well-understood structures
- Complexity can always be added, but this needs to be done in a disciplined way
- Otherwise we would have to rely immediately on numerical methods which are routinely used for large-scale macroeconomic models
- But such methods will only be illuminating if the core of a model is sufficiently simple such that it can be ‘understood’
- Subsequent lectures will cover extensions and add additional features
I Motivation

The basic model of the centralized economy, notwithstanding its simplicity, has been very influential over decades.

→ **Interpretations of the basic model:**

- Frank **Ramsey** (1927) introduced a similar version to study taxation issues. Hence, the model is often called the **Ramsey model**.

- The model can be interpreted as a **social planning model** in which decisions are taken by the central planner, taking as given individual preferences.

- The model gives rise to a **representative agent model**, in the sense that all economic agents are identical (and households and firms have the same objectives).

- Since there exists, in fact, only a single individual, the model describes a **Robinson Crusoe economy**.

- The model is the basis of **neoclassical growth theory** (Solow, 1956, Cass, 1965, Koopmans, 1967).
Consider a closed economy with a constant population $N$.

In a representative period $t$, we consider the following aggregate variables (using capital letters):
- $Y_t$: output
- $C_t$: consumption
- $K_t$: predetermined level of capital available for production
- $I_t$: gross investment undertaken within the period
- $S_t$: savings

Alternatively, consider these variables in per capita form (using lower case letters), i.e., output per capita is given by

$$y_t = \frac{Y_t}{N}$$

Similarly:

$$c_t = \frac{C_t}{N}, \quad k_t = \frac{K_t}{N}, \quad i_t = \frac{I_t}{N}, \quad s_t = \frac{S_t}{N}$$
II Basic model ingredients

Key equations

To capture choices between ‘consumption today’ and ‘consumption tomorrow’ in a closed economy consider 3 basic equations (per capita form)

1) Resource constraint (national income identity):

\[ y_t = c_t + i_t, \]  

where we use that savings are equal to investment, ie

\[ s_t = y_t - c_t = i_t \]

2) Capital stock dynamics

\[ \Delta k_{t+1} = \underbrace{i_t - \delta k_t}_{k_{t+1} - k_t} \]

saying that \( \Delta k_{t+1} \) (ie net investment) results from gross investment \( i_t \) minus depreciation (where we assume that a constant proportion \( \delta \in (0, 1) \) of the existing capital stock depreciates in period \( t \))
II Basic model ingredients

Key equations

3) Production function

\[ y_t = f(k_t) \]  \hspace{1cm} (3)

Idea: The ‘neoclassical’ production function \( f \) is such that an increase in \( k \) increases output, but at a diminishing rate.

Let \( k > 0 \). Then:

\[ f(k) > 0, \quad f'(k) > 0, \quad f''(k) < 0 \]

Moreover:

\[ \lim_{k \to 0} f'(k) \to \infty, \quad \lim_{k \to \infty} f'(k) \to 0 \]

These are the so-called ‘Inada-conditions’. What do they say?

- at the origin there are infinite output gains to increasing \( k \)
- these gains decline as \( k \) becomes larger
- they eventually disappear if \( k \) becomes arbitrarily large
Comment: Production function (aggregate vs. per capita output)

- Notice that
  
  \[ y_t = f(k_t) \]

  is in per capita form

- The aggregate production is given by

  \[ Y_t = F(K_t, N). \]

In neoclassical tradition, \( F \) has **constant returns to scale**, i.e., for any proportionate variation \( \lambda \) of both inputs the function \( F \) satisfies

\[ F(\lambda K_t, \lambda N) = \lambda F(K_t, N) = \lambda Y_t \]

- Hence, assuming \( \lambda = \frac{1}{N} \), per capita output satisfies

  \[ y_t = \frac{Y_t}{N} = F(k_t, 1) \equiv f(k_t) \]

- ‘**Notice**’: In some textbooks (e.g., Wickens) you find the alternative notation for per capita output

  \[ F(k_t, 1) \equiv F(k_t) \]
II Basic model ingredients

Key equations

- We can combine eqns (1)-(3) and eliminate $y_t$ and $i_t$ such that the resource constraint simplifies to

$$f(k_t) = c_t + \Delta k_{t+1} + \delta k_t$$

- Since $\Delta k_{t+1} = k_{t+1} - k_t$, this equation acts like a dynamic constraint on the economy.

- Equivalently, to see how this equation restricts the feasible choices of consumption over time, write it as

$$c_t = f(k_t) - k_{t+1} + (1 - \delta)k_t$$  \hspace{1cm} (4)
Interpretation:

- Eqn (4), i.e.

\[ c_t = f(k_t) - k_{t+1} + (1 - \delta)k_t \]

can be read as follows:
- Consider an initial period \( t = 0 \) with a given (i.e. predetermined) value \( k_0 \) (which fixes output \( f(k_0) \) in period \( t = 0 \))
- Assume there exists some rule or some regularity which tells us for the given value of \( k_0 \) how to determine the consumption level \( c_0 \). This will implicitly determine \( k_1 \).
- If we use the same rule again in \( t = 1 \) we find \( c_1 \), and, implicitly, \( k_2 \)
- Continuing this recursive logic for \( t = 2, 3, \ldots, T \), we can derive the entire sequence of \( c \) and \( k \) into the infinite future (i.e. \( T \to \infty \))

- Notice that eqn (4) is non-linear because of the term \( f(k_t) \)
II Basic model ingredients
Possible choices for consumption: overview

Given the just derived dynamic constraint (4), ie

$$c_t = f(k_t) - k_{t+1} + (1 - \delta)k_t$$

we need some criterion or objective in order to determine optimal choices of consumption

- An extreme choice would be entirely myopic, ie for a given value $k_0$ the highest possible level of $c_0$ in period $t = 0$ amounts to

$$c_0^{myopic} = f(k_0) + (1 - \delta)k_0$$

Yet, this choice would imply $k_1 = 0$, ie it is not sustainable (in fact, it would imply zero output and zero consumption in all future periods!)
A more reasonable criterion is to impose that consumption levels should be sustainable, i.e., consumption should be maximized in each period.

We will consider two alternatives: the so-called golden rule solution and an optimal solution.

The key difference between the two solution concepts is that under the optimal solution future consumption will be discounted, while the golden rule ignores discounting.
The golden rule solution is derived from a long-term objective: it maximizes the (constant) amount of per capita consumption in each period by doing so, it treats members of different generation alike ('golden rule')

Hence, going back to eqn (4), ie

\[ c_t = f(k_t) - k_{t+1} + (1 - \delta)k_t \]

we consider a long-run (or ‘steady-state’) situation in which all per capita variables are constant (in particular \( k_t = k_{t+1} = k \)), leading to

\[ c = f(k) - \delta k \] (5)

Eqn (5) implies that net investment will be zero, ie the only investment undertaken is such that it replaces depreciated capital, facilitating a constant capital stock over time
Given the steady-state resource constraint (5), ie
\[ c = f(k) - \delta k, \]
how should one optimally choose \( c \)?

To find the golden rule solution we solve the maximization problem
\[
\max_k f(k) - \delta k
\]

The golden rule capital stock \( k_{GR} \) is implicitly characterized by the first-order condition
\[
\frac{dc}{dk} = f'(k_{GR}) - \delta = 0,
\]
and the second-order condition, evaluated at \( k_{GR} \),
\[
\frac{d^2c}{dk^2} = f''(k_{GR}) < 0
\]
ensures that \( k_{GR} \) is a maximum (and not a minimum).
Uniqueness:
Given the assumptions on $f$, the optimum $k_{GR}$ which solves

$$f'(k_{GR}) = \delta$$

is unique and the associated unique consumption level $c_{GR}$ is given by

$$c_{GR} = f(k_{GR}) - \delta k_{GR}$$

Interpretation of the golden rule solution:

- Eqn (6) says that steady-state per capita consumption will be maximized if the marginal product of $k$ equals the depreciation rate $\delta$

- Below the level $k_{GR}$ a marginal increase in $k$ increases $c$, since the marginal gain in output (ie $f'(k)$) exceeds the output cost of replacing depreciated capital

- Above the level $k_{GR}$ a marginal increase in $k$ would decrease $c$, since the marginal gain in output (ie $f'(k)$) is smaller than the output cost of replacing depreciated capital
Let us use the golden rule solution to introduce the notion of *comparative statics*:

- **Idea**: how do long-run (steady-state) solutions of endogenous variables change if an exogenous parameter changes?
- **Typically** we can sign these changes, by using the information embodied in the functional forms that are used.

**Particular example:**

→ Assume the rate $\delta$ at which capital depreciates increases...

→ ...How do $k_{GR}$ and $c_{GR}$ react to the exogenous change in $\delta$?
III Golden rule solution
Comparative statics

**Particular comparative statics example: increase in \( \delta \)**

- Recall that the first-order optimality condition
  \[
  f'(k_{GR}) = \delta
  \]
  establishes only an **implicit** dependence of \( k_{GR} \) on \( \delta \), ie we cannot directly differentiate \( k_{GR} \) with respect to \( \delta \).

- Yet, since this optimality condition will be satisfied for any exogenous value \( \delta \), we can write it as an identity
  \[
  f'(k_{GR}(\delta)) - \delta \equiv 0
  \]
  (8)

- Differentiating (8) w.r.t. \( \delta \) (where we use the chain rule) yields
  \[
  f''(k_{GR}) \cdot \frac{dk_{GR}}{d\delta} - 1 \equiv 0,
  \]
  implying
  \[
  \frac{dk_{GR}}{d\delta} = \frac{1}{f''(k_{GR})} < 0
  \]
  (9)

  \( \rightarrow \) an increase in \( \delta \) makes the accumulation of capital more costly, leading to a decline in \( k_{GR} \)
What about the reaction of \( c_{GR} \) to a change in \( \delta \)?

- To respect the implicit dependence of \( k_{GR} \) on \( \delta \), express (7) as

\[
c_{GR} = f(k_{GR}(\delta)) - \delta k_{GR}(\delta)
\]

- Differentiating \( c_{GR} \) with respect to \( \delta \) gives:

\[
\frac{dc_{GR}}{d\delta} = \frac{d[f(k_{GR}(\delta)) - \delta k_{GR}(\delta)]}{d\delta}
\]

\[
= \left[ f'(k_{GR}) - \delta \right] \frac{dk_{GR}}{d\delta} - k_{GR}(\delta)
\]

\[
= \begin{cases} 
0 & \text{if } f'(k_{GR}) = \delta \\
- k_{GR}(\delta) < 0 & \text{otherwise}
\end{cases}
\]

\( \rightarrow \) an increase in \( \delta \) leads also to a decline in \( c_{GR} \)
Comment:

- To derive comparative statics results from implicit relationships like
  \[ f'(k_{GR}) = \delta \]
  there exist alternative techniques
- In particular, if one **totally differentiates** the relationship at the equilibrium one obtains
  \[ f''(k_{GR}) \cdot dk = d\delta, \]
  which can be rearranged to confirm (9), ie
  \[ \frac{dk_{GR}}{d\delta} = \frac{1}{f''(k_{GR})} < 0. \]
Lecture 1 argued that modern macroeconomics attempts to base the analysis on micro-founded welfare criteria, consistent with optimizing behaviour of the representative consumer.

The golden rule analysis carefully incorporates the dynamic constraint relating to capital stock dynamics...

...but it is silent on whether there exists an individual welfare measure that would generate the golden rule solution.
In particular, the golden rule analysis pretends that individuals value consumption today and consumption tomorrow in the same way.

But this is not a satisfactory assumption, given the observed impatience in decisions of consumers.

This aspect is captured by the so-called optimal solution (meaning that the optimality criterion corresponds to a micro-founded welfare objective which incorporates impatience).
IV Optimal solution

Objective

- Let the representative period be denoted by $t$
- Assume there exists in the initial period $t = 0$ a predetermined per capita capital stock $k_0$
- Let $V_0$ denote the present value of current and future utility, as given by:

$$V_0 = \sum_{t=0}^{\infty} \beta^t U(c_t), \quad (10)$$

where the instantaneous utility $U_t = U(c_t)$ satisfies $U'(c_t) > 0$ and $U''(c_t) < 0$, i.e., within any period additional consumption increases utility but at a diminishing rate.
- The objective $V_0$ is additively separable which makes it easy to compare utility between periods.
- Future utility is discounted by the constant factor $\beta$ which satisfies $0 < \beta < 1$.
- Alternatively, we can define the corresponding discount rate $\theta > 0$, with:

$$\beta = \frac{1}{1 + \theta}$$
IV Optimal solution

Objective

- The goal pursued by the optimal solution is to choose current and future consumption such that the objective (10), ie

\[ V_0 = \sum_{t=0}^{\infty} \beta^t U(c_t), \]

will be maximized subject to the above established dynamic constraint (4), ie

\[ c_t = f(k_t) - k_{t+1} + (1 - \delta)k_t \]

- We will solve this dynamic optimization problem by using the Lagrange multiplier technique
IV Optimal solution
Solution based on Lagrange multipliers

→ Consider the objective $L$ which incorporates the resource constraint (4)
→ In order to maximize (10) s.t. (4) we optimize, equivalently,

$$L = \sum_{t=0}^{\infty} \{ \beta^t U(c_t) + \lambda_t [f(k_t) - c_t - k_{t+1} + (1 - \delta)k_t] \}$$

over the choice variables $\{c_t, k_{t+1}, \lambda_t; \forall t \geq 0\}$
→ $\lambda_t$ is the Lagrange multiplier $t$ periods ahead, measuring the shadow value of an additional unit of period $t$ income (in terms of utility of period 0)

**First-order optimality conditions** (‘FOCs’, interior) w.r.t. $c_t, k_t, \lambda_t$ :

$$\frac{\partial L}{\partial c_t} = \beta^t U'(c_t) - \lambda_t = 0 \quad t \geq 0 \quad (11)$$

$$\frac{\partial L}{\partial k_t} = \lambda_t [f'(k_t) + (1 - \delta)] - \lambda_{t-1} = 0 \quad t > 0 \quad (12)$$

$$\frac{\partial L}{\partial \lambda_t} = f(k_t) - c_t - k_{t+1} + (1 - \delta)k_t = 0 \quad t \geq 0 \quad (13)$$

**Transversality condition:**

$$\lim_{t \to \infty} \beta^t \cdot U'(c_t) \cdot k_{t+1} = 0$$

(14)
Comment: How to read the just derived equations (11)-(14)?

- These are **necessary conditions for optimality**
- The sufficient conditions for a maximum are satisfied, given our assumptions on functional forms

*Notice:* The concept of intertemporal optimality applies to sequences of variables, i.e., the equations form a **system of difference equations** characterizing the behaviour of the equilibrium over time

- Crucial for the exact time paths of variables consistent with such system: **initial** and **terminal** conditions
Comment: Initial condition

- By assumption, the economy starts to operate in $t = 0$, taken as given the predetermined level of the per capita capital stock $k_0$
  $\rightarrow k$ is the single **predetermined (state) variable** of the system

- In period $t = 0$, the per capita consumption level $c_0$ can be freely chosen
  $\rightarrow c$ is the single **forwardlooking (control) variable** w/o initial condition

- These features will become relevant when we discuss stability issues below
Comment: Terminal condition

- The **transversality condition** (14), ie

\[
\lim_{t \to \infty} \beta^t \cdot U'(c_t) \cdot k_{t+1} \cdot \lambda_t = 0,
\]

...closes the system by backward induction from the (distant) future...

- To see how this can be made operational, consider first some large and finite value of \( t \), ie a distant period somewhere far out in the future...
Comment: Terminal condition

- For any finite value of $t$, the term

$$\beta^t \cdot U'(c_t) \cdot k_{t+1} = \lambda_t \cdot k_{t+1}$$

describes the present value of the utility that could be obtained if $k_{t+1}$ (i.e., the capital stock for the next period resulting from investment decisions in $t$) will be consumed at $t$ rather than being left for production for $t+1$.

- If this particular value of $t$ marks the terminal period it cannot be optimal, not to consume everything in the terminal period.

- **Infinite horizon analogy**: There exists no terminal period, but as $t \to \infty$, it cannot be optimal to postpone consumption forever, i.e.

$$\lim_{t \to \infty} \lambda_t k_{t+1} = 0,$$

as specified by (14).
IV Optimal solution
Solution based on Lagrange multipliers

Simplification of the FOCs:

- Let us reconsider the FOCs (11) and (12), ie
  \[ \beta^t U'(c_t) - \lambda_t = 0 \quad t \geq 0 \]
  \[ \lambda_t [f'(k_t) + (1 - \delta)] - \lambda_{t-1} = 0 \quad t > 0 \]

- We can obtain the Lagrange multiplier from the first eqn and substitute for \( \lambda_t \) and \( \lambda_{t-1} \), respectively, in the second eqn, leading to
  \[ \beta^t U'(c_t) [f'(k_t) + (1 - \delta)] = \beta^{t-1} U'(c_{t-1}) \quad t > 0 \]

- Equivalently, after dividing by \( \beta^{t-1} \) and updating of all terms by one period, we can rewrite this eqn as
  \[ \beta U'(c_{t+1}) [f'(k_{t+1}) + (1 - \delta)] = U'(c_t), \quad t \geq 0 \quad (15) \]

which is the so-called consumption Euler equation
IV Optimal solution
What do we get?

2 key equations:

- Recall from above that via eqn (13) the optimization preserved the dynamic resource constraint (4)

- In sum, the (consolidated) intertemporal equilibrium consists of the consumption Euler equation and the resource constraint, ie we have $\forall t \geq 0$:

  \[
  U'(c_t) = \beta U'(c_{t+1})[f'(k_{t+1}) + (1 - \delta)] \\
  c_t = f(k_t) - k_{t+1} + (1 - \delta)k_t
  \]

  → We have reduced the dynamics to a non-linear two-dimensional dynamic system in $c$ and $k$ with one initial condition ($k_0$) and one terminal condition, as given by the transversality condition (14)

- Before we analyze the system (16)-(17), we will give some more interpretation to the consumption Euler equation
IV Optimal solution
What do we get?

Interpretation of the consumption Euler equation:

- The consumption Euler equation (16), ie

\[
\beta U'(c_{t+1})[f'(k_{t+1}) + (1 - \delta)] = U'(c_t), \quad t \geq 0
\]

is the fundamental dynamic equation in intertemporal optimization problems in which consumers actively decide about how to choose between ‘consumption today’ and ‘consumption tomorrow’

- In eqn (16), ‘today’ corresponds to \( t = 0 \). Since the optimization holds for all \( t \geq 0 \), the recursive nature of the FOCs implies that ‘tomorrow’ covers not only \( t = 1 \), but all subsequent future periods, ie \( t = 2, 3, \ldots T \ldots \text{etc.} \)
IV Optimal solution
What do we get?

- The consumption Euler equation

\[ \beta U'(c_{t+1})[f'(k_{t+1}) + (1 - \delta)] = U'(c_t) \]

can be seen as an **intertemporal arbitrage condition**, saying that at the optimum the representative consumer must be indifferent between consuming a marginal unit of \( c \), yielding extra utility \( U'(c_t) \),

or, alternatively, investing this unit and consuming the return one period later, yielding extra utility

\[ \beta U'(c_{t+1})[f'(k_{t+1}) + (1 - \delta)] \]

- The discount factor \( \beta \) ensures that consumption today and tomorrow will be comparable in terms of utility
IV Optimal solution
Long-run (steady-state) features of the optimal solution

- Let us go back to the pair of equilibrium eqns (16) and (17), ie

\[
\begin{align*}
U'(c_t) &= \beta U'(c_{t+1})[f'(k_{t+1}) + (1 - \delta)] \\
c_t &= f(k_t) - k_{t+1} + (1 - \delta)k_t
\end{align*}
\]

- **Long-run (‘steady-state’) equilibria** exhibit constant variables

- From (16), the optimal long-run (per capita) levels \( k^* \) and \( c^* \) satisfy

\[
U'(c^*) = \beta U'(c^*)[f'(k^*) + (1 - \delta)],
\]

implying

\[
f'(k^*) = \frac{1}{\beta} - 1 + \delta = \delta + \theta \quad (18)
\]

- From (17):

\[
c^* = f(k^*) - \delta k^* \quad (19)
\]
IV Optimal solution
Long-run (steady-state) features of the optimal solution

→ Steady states of the optimal solution satisfy (18) and (19), ie

\[ f'(k^*) = \delta + \theta \]
\[ c^* = f(k^*) - \delta k^* \]

- Eqns (18) and (19) can be solved sequentially for \( k^* \) and \( c^* \)
- Given the assumptions on \( f \), there exists a unique steady state

Interpretation of the (steady-state) optimal solution:

- The optimal solution has \( k^* < k_{GR} \), since \( \delta + \theta > \delta \)
- Moreover, \( c^* < c_{GR} \), since \( c^* \) does not maximize \( f(k) - \delta k \)
- These findings reflect the role of \( \theta \): because of impatience the representative consumer does not reach the higher long-run consumption level \( c_{GR} \)
Recall from above that dynamics are governed by eqns (16) and (17), ie

\[ U'(c_t) = \beta U'(c_{t+1})[f'(k_{t+1}) + (1 - \delta)] \]
\[ c_t = f(k_t) - k_{t+1} + (1 - \delta)k_t, \]

ie a non-linear two-dimensional dynamic system in \( c \) and \( k \) with one initial condition \( (k_0) \) and one terminal condition (ie the TV-condition (14))

It can be shown that the dynamics are stable in a particular sense, ie the system is (locally) saddlepath-stable
IV Optimal solution

Dynamics of the optimal solution

**Saddlepath-stability:**

- **Saddlepath-stability** means that for any value $k_0$ close to the long-run value $k^*$ there exists a unique value $c_0$ which
  i) satisfies all optimality conditions and
  ii) sets in motion sequences $\{c_t, k_{t+1}\}_{t=0}^{t=\infty}$ that ultimately converge against the long-run values $c^*$ and $k^*$

- To calculate analytically the saddlepath requires some knowledge of matrix algebra...

- ...but the saddlepath-stable behaviour can be illustrated with a **phase diagram** which summarizes the dynamic forces of a linearized version of eqns (16) and (17)
Consider eqn (16), which displays non-linear dynamics in $c$, ie

$$U'(c_t) = \beta U'(c_{t+1})[f'(k_{t+1}) + (1 - \delta)]$$

To obtain locally linear dynamics in $c$, approximate $U'(c_{t+1})$, using a first-order Taylor expansion around the point $c_t$, such that

$$U'(c_{t+1}) \approx U'(c_t) + U''(c_t) \cdot \Delta c_{t+1} \quad \Leftrightarrow \quad \frac{U'(c_{t+1})}{U'(c_t)} \approx 1 + \frac{U''(c_t)}{U'(c_t)} \cdot \Delta c_{t+1}$$

Use this approximation in eqn (16) to get

$$1 + \frac{U''(c_t)}{U'(c_t)} \cdot \Delta c_{t+1} = \frac{1}{\beta[f'(k_{t+1}) + (1 - \delta)]}$$

$$\Delta c_{t+1} = -\frac{U'(c_t)}{U''(c_t)} \cdot \left[1 - \frac{1}{\beta[f'(k_{t+1}) + (1 - \delta)]}\right]$$

(20)
The phase diagram will be organized around eqns (17) and (20), ie

\[
\Delta k_{t+1} = f(k_t) - \delta k_t - c_t \\
\Delta c_{t+1} = -\frac{U'(c_t)}{U''(c_t)} \cdot \left[ 1 - \frac{1}{\beta[f'(k_{t+1}) + (1 - \delta)]} \right]
\]

Notice that if \( c_t = c^* \) and \( k_t = k^* \) then \( \Delta k_{t+1} = \Delta c_{t+1} = 0 \)

Dynamic implication of eqn (17): it features no dynamics in \( c \), only in \( k \) such that

\( \Delta k_{t+1} \leq 0 \) if \( c_t \geq f(k_t) - \delta k_t \)

Dynamic implication of eqn (20): it features no dynamics in \( k \), only in \( c \) such that

\( \Delta c_{t+1} \leq 0 \) if \( k_{t+1} \geq k^* \)

These informations can be combined to represent the dynamics in \( c_t \) and \( k_t \) via a phase diagram.
IV Optimal solution
Dynamics of the optimal solution

Comments on the phase diagram of the linearized dynamics in $c_t$ and $k_t$

- Arrows indicate regions of stability and instability around $k^* > 0$, $c^* > 0$
- For any initial departure of the state variable such that $k_0 \neq k^*$: Saddlepath configuration, i.e. there exists a unique choice of the control variable $c_0$ such that the economy ‘jumps’ on the saddlepath and converges over time towards the steady state $k^*$, $c^*$

How does consumption optimally react along the saddlepath?

i) Consider a temporary negative shock to the capital stock: $k_0 < k^*$. → The saddlepath is such that on impact $c_0 < c^*$ will be optimal → Thus, temporarily, consumption will be smaller than $c^*$ such that some output can be diverted to rebuild the capital stock → This flexible short-run response of consumption is optimal, since it ensures that the long-run level $c^*$ remains feasible

ii) Consider a temporary positive shock to the capital stock: $k_0 > k^*$. → The reverse response pattern will be optimal, i.e. $c_0 > c^*$ → Temporarily, consumption can be larger than $c^*$, w/o endangering the feasibility of the long-run level $c^*$
IV Optimal solution
Dynamics of the optimal solution

Comments on the phase diagram of the linearized dynamics in $c_t$ and $k_t$

- Important information not yet used: (i) $k \geq 0$, and (ii) TV-condition (14)
  → For all other choices of $c_0$ (i.e., off the saddlepath), the dynamics ultimately drift away from $k^*, c^*$

- Such choices can be ruled out because the economy would eventually hit either:
  a **path of rising consumption and falling capital** on which $k$ would become negative (but this cannot be)
  or:
  a **path of falling consumption and rising capital** on which the present value of lifetime consumption would become smaller than the present value of lifetime income (but this cannot be optimal)

- In sum, saddlepath-stability implies that the system is not only stable, but that the dynamics towards the steady state are uniquely determined