Mathematical Methods, Part 1:
Applied Intertemporal Optimization

Part II

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Part II

Stochastic Models in Discrete Time
3 Setting up a random environment

• In a stochastic world, all quantities take the form of random variables
• We will first review some basic concepts from probability theory required for our purposes
• Following structure is very condensed. You are strongly encouraged to consult Stachurski (2009) for a more in-depth treatment.
• In the sequel we work with infinite discrete time periods \( T = \{0, 1, 2, \ldots\} \).
• If \( A \) is any set, \( 2^A \) or \( \text{Pow}(A) \) denotes the power set, i.e., the class of all subsets of \( A \).
3.1 Basic concepts from probability theory

3.1.1 Probability spaces and random variables

- Randomness in our model enters via an exogenous stochastic process \((\theta_t)_{t \geq 0}\), i.e., a sequence of random variables with values in \(\Theta \subset \mathbb{R}^N, N \geq 1\).

- All these random variables live on an underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where:
  - \(\Omega\) is the sample space which represents all possible states of the world
  - \(\mathcal{F} \subset \text{Pow}(\Omega)\) is a collection of subsets of \(\Omega\) that form a \(\sigma\)-algebra, i.e., (i) \(\Omega \in \mathcal{F}\), (ii) \(A \in \mathcal{F}\) implies \(A^c := \Omega \setminus A \in \mathcal{F}\) and (iii) \((A_n)_{n \geq 0}, A_n \in \mathcal{F}\) \(\forall n\) implies \(\bigcup_{n=0}^{\infty} A_n \in \mathcal{F}\).
  - \(\mathbb{P} : \mathcal{F} \rightarrow [0, 1]\) is a probability measure, i.e., a countably additive function satisfying \(\mathbb{P}(\Omega) = 1\) that assigns probabilities \(\mathbb{P}(A)\) to each measurable subset \(A \in \mathcal{F}\) of \(\Omega\).

- \(\Theta\) is endowed with some \(\sigma\)-algebra \(\mathcal{A} \subset \text{Pow}(\Theta)\) to become a measurable space \((\Theta, \mathcal{A})\).

- Since \(\Theta \subset \mathbb{R}^N\) is a topological space, we can (and typically do) choose for \(\mathcal{A}\) the Borel-\(\sigma\) algebra \(\mathcal{B}(\Theta)\) which is the smallest \(\sigma\)-algebra containing the topology.

- For each \(t \in \mathbb{T}\), the mapping \(\theta_t : \Omega \rightarrow \Theta\) is \(\mathcal{F} - \mathcal{B}(\Theta)\) measurable, i.e., for all \(B \in \mathcal{B}(\Theta)\), \(\theta_t^{-1}(B) := \{\omega \in \Omega | \theta_t(\omega) \in B\} \subset \Omega\) is an element of \(\mathcal{F}\).
3.1.2 Probability and distributions of random variables

- For each $t \in \mathbb{T}$, can construct probability distribution/measure $\mu_t$ of random variable $\theta_t$:
  - given a set $B \in \mathcal{B}(\Theta)$, $\mu_t(B)$ is the probability that $\theta_t \in B$
  - straightforward to construct $\mu_t$ by defining the image measure
    \[
    \mu_t(B) = \mathbb{P}(\theta_t^{-1}(B)) = \mathbb{P}\left(\{\omega \in \Omega | \theta_t(\omega) \in B\}\right) \quad (81)
    \]
  - mapping $\mu_t : \mathcal{B}(\Theta) \longrightarrow [0, 1]$ is indeed a probability measure on $(\Theta, \mathcal{B}(\Theta))$ and called the probability distribution of $\theta_t$
  - if $\Theta = \mathbb{R}$, there is a one-to-one correspondence between distribution $\mu_t$ and the distribution function $F_t(b) := \mu_t(\mathbb{R}^{-\infty}, b)$, $b \in \mathbb{R}$. Similar result holds if $N > 1$.

- Analogously, construct joint distribution $\mu_{\mathbb{I}}$ of random variables $\theta_{\mathbb{I}} := (\theta_t)_{t \in \mathbb{I}}$ for any $\mathbb{I} \subset \mathbb{T}$

- Further, can infer the distributions of random variables defined by measurable functions
  - $f : \Theta \longrightarrow X \subset \mathbb{R}^M$ of $\theta_t$
  - $f : \Theta^\mathbb{I} \longrightarrow X \subset \mathbb{R}^M$ of $\theta_{\mathbb{I}}$

  with values in the measurable space $(X, \mathcal{B}(X))$
3.2 Constructing the underlying probability space

- Previous result: Given \((\Omega, \mathcal{F}, P)\) and measurable mappings \((\theta_t)_t\), can compute probability distributions of all random variables \((\theta_t)_{t \in I}, I \subset T\) and of all measurable functions of these random variables.

- Can also reverse the previous construction:
  - specify distributions/dependence structure of the random variables \((\theta_t)_{t \in T}\)
  - construct an underlying probability space \((\Omega, \mathcal{F}, P)\) consistent with this.

3.2.1 Example 1: Independent random variables

- Suppose we want \((\theta_t)_{t \geq 0}\) to consist of independent random variables with values in \(\Theta\) each having a desired probability distribution \(\mu : \mathcal{B}(\Theta) \rightarrow [0, 1]\), say, a normal distribution.

- In this case, define:
  - \(\Omega = \Theta^T\) (the set of sequences with values in \(\Theta\))
  - \(\mathcal{F} = \mathcal{B}(\Omega)\) (the product \(\sigma\)-algebra generated by measurable rectangles or, equivalently, the Borel \(\sigma\)-algebra when \(\Omega\) is endowed with the product topology)
  - \(P = \mu^T\) (the product measure which satisfies \(\mu^T(\Omega \times \ldots \times \Omega \times A \times B \times \Omega \times \ldots) = \mu(A) \cdot \mu(B)\) for any \(A, B \in \mathcal{B}(\Theta)\))
3.2.2 Example 2: Correlated random variables

- Suppose we want \((\theta_t)_{t \in \mathbb{T}}\) to follow an auto-regressive structure of the form

\[ \theta_t = M\theta_{t-1} + \varepsilon_t, \quad t \geq 1, \]  

(82)

where \(M \in \mathbb{R}^{N \times N}\) and \((\varepsilon_t)_{t \geq 1}\) consists of i.i.d. random variables with values in \(\mathcal{E} \subset \mathbb{R}^N\) and distribution \(\mu_\varepsilon\) which are independent of \(\theta_0\) which has distribution \(\mu_0\).

- In this case, can also construct \((\Omega, \mathcal{F}, \mathbb{P})\) by defining \(\Omega = \Theta \times \mathcal{E}^N\), \(\mathcal{F} = \mathcal{B}(\Omega)\), \(\mathbb{P} = \mu_0 \otimes \mu_\varepsilon^N\).

- Noting that \(\theta_t = A^t\theta_0 + \sum_{n=0}^{t-1} M^n \varepsilon_{t-n}\) we can compute \(\mu_t\) for each \(t > 0\) via (81).

- For later reference, note that (121) defines a transition probability, i.e., a mapping \(Q : \Theta \times \mathcal{B}(\Theta) \rightarrow [0,1]\) such that \(Q(\theta, A)\) is the probability that \(\theta_{t+1} \in A\) given that \(\theta_t = \theta\).

- For all \(\theta \in \Theta\) and \(A \in \mathcal{B}(\Theta)\), \(Q\) can explicitly be constructed as

\[ Q(\theta, A) = \mu_\varepsilon\{\varepsilon \in \mathcal{E} | M\theta + \varepsilon \in A\} \]  

(83)

- The distributions \((\mu_t)_{t \in \mathbb{T}}\) can then be computed recursively for \(t \geq 1\) as

\[ \mu_t(B) = \int_{\Theta} Q(\theta, B) \mu_{t-1}(d\theta). \]  

(84)

for each \(B \in \mathcal{B}(\Theta)\).
3.3 Filtration and conditional expectation

- Let \((\theta_t)_{t \geq 0}\) be the exogenous stochastic process on \((\Omega, \mathcal{F}, \mathbb{P})\) defined previously.

- In our equilibrium framework derived below, all endogenous variables will take the form of random variables \((X_t)_{t \geq 0}\) with values in \(X \subset \mathbb{R}^M\) which depend on the exogenous process.

- We generally take the notation \((X_t)_{t \geq 0}\) to mean that \(X_t\) is observable in period \(t\), i.e., can only depend on exogenous random variables \(\theta_n, n \leq t\).

- To impose this restriction formally, define a filtration \((\mathcal{F}_t)_{t \geq 0}\) where \(\mathcal{F}_t \subset \mathcal{F}\) is the smallest \(\sigma\)-algebra such that each \(\theta_n, 0 \leq n \leq t\) is \(\mathcal{F}_t\)-\(\mathcal{B}(\Theta)\) measurable.

- Process \((X_t)_{t \in \mathbb{T}}\) is said to be adapted (to \((\mathcal{F}_t)_{t \geq 0}\)) if each \(X_t\) is \(\mathcal{F}_t - \mathcal{B}(X)\) measurable. This captures exactly the idea that \(X_t\) can depend only on random variables \(\theta_n, n \leq t\).

- Specifically, if \((X_t)_{t \in \mathbb{T}}\) is adapted and \(\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_t]\) is the expectations operator conditional on observations up to time \(t\), \(\mathbb{E}_t[X_n] = X_n\) for all \(t\) and \(n \leq t\).

- If \(X_t\) has distribution \(\mu_{X_t} : \mathcal{B}(\mathbb{X}) \rightarrow [0, 1]\) and is integrable, its unconditional expectation is defined as

\[
\mathbb{E}[X_t] := \int_{\Omega} X_t(\omega) \mathbb{P}(d\omega) = \int_{\mathbb{X}} x \mu_{X_t}(dx). \tag{85}
\]
4 Stochastic decision problems with finite horizon

4.1 A stochastic OLG model

- Consider a stochastic version of the OLG model from Section 1.6 similar to Wang (1993):
  - all assumptions on population structure, labor supply, etc. remain the same
  - but: production side modified to incorporate random production shocks

- We continue to denote equilibrium variables as \((X_t)_{t \geq 0}\) but these are now adapted stochastic processes rather than just sequences.

- All equalities and inequalities involving random variables are assumed to hold \(\mathbb{P}\)-almost surely without explicit notice.
4.1.1 Production side

- Suppose that production in period $t$ is subject to multiplicative shock $\theta_t \in \Theta \subset \mathbb{R}^++$:
  \[ Y_t = \theta_t F(K_t, L_t) = \theta_t L_t f(k_t) \] (86)

- Production shocks $(\theta_t)_{t \geq 0}$ consists of independent random variables with distribution $\mu$ and values in $\Theta = [\theta_{\text{min}}, \theta_{\text{max}}] \subset \mathbb{R}^++$.

- Thus, we can chose the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ exactly as outlined in Section 3.2.1

- In each period $t$, the firm takes the current shock in period $t$ as given and decides on demand for capital and labor.

- Continue to impose Assumption 1.3 on $f$ and define $k_t = \frac{K_t}{L_t}$ for all $t \in \mathbb{T}$ as before.

- The first order conditions then determine equilibrium factor prices as:
  \[ w_t = \mathcal{W}(k_t, \theta_t) := \theta_t [f(k_t) - k_t f'(k_t)] \] (87a)
  \[ R_t = \mathcal{R}(k_t, \theta_t) := \theta_t f'(k_t) \] (87b)
4.1.2 A stochastic two-period decision problem

- Consider decision problem of a young consumer in period $t \geq 0$:
  - consumer observes her current labor income $w_t > 0$ (which is a real number)
  - capital return $R_{t+1}$ treated as random variable with values in $[R_{\text{min}}, R_{\text{max}}] \subset \mathbb{R}_{++}$
  - knowing the underlying probabilistic structure, consumer computes correct conditional expectation $\mathbb{E}_t[\cdot]$ of next period’s random variables

- Any investment decision $s_t \in [0, w_t]$ (which is a number!) determines lifetime consumption
  \[ c^y_t = w_t - s_t \]
  \[ c^o_{t+1} = R_{t+1}s_t \]

  where $c^y_t \geq 0$ is a number and $c^o_{t+1}$ a random variable with values in $[s_t R_{\text{min}}, s_t R_{\text{max}}]$.

- Preferences over alternative random variables $(c^y_t, c^o_{t+1})$ possess an expected utility representation with von-Nemann Morgenstern utility $U(c^y, c^o) = u(c^y) + \beta u(c^o)$

- Decision problem reads:
  \[
  \max_s \left\{ u(w_t - s) + \beta \mathbb{E}_t [u(s R_{t+1})] \mid 0 \leq s \leq w_t \right\}
  \]
• Define consumer’s objective function $U_t : [0, w_t] \rightarrow \mathbb{R}$,

$$U_t(s) := u(w_t - s) + \beta \mathbb{E}_t[u(sR_{t+1})]$$  \hspace{1cm} (90)

• Imposing Assumption 1.2 on utility $u$, we obtain the following result:

**Lemma 4.1** Under Assumption 1.2, the following holds:

(i) $U_t$ in (90) is $C^2$, strictly concave, and $\lim_{s \to 0} U_t'(s) = -\lim_{s \to w_t} U_t'(s) = -\infty$

(ii) Problem (89) has a unique interior solution $s_t^*$ determined by

$$u'(w_t - s) = \beta \mathbb{E}_t[R_{t+1}u'(sR_{t+1})]$$  \hspace{1cm} (91)

• **Hint:** When proving this result, exploit that in the present case, differentiation can be interchanged with the expectations operator
4.1.3 Deriving the equilibrium equations

- Aggregate investment made at time $t$ determines next period's capital stock $K_{t+1} = Ns_t$
- Defining $k_{t+1} = K_{t+1}/N$, (91) can be written as:
  \[ u'(w_t - k_{t+1}) = \beta \mathbb{E}_t [R_{t+1}u'(k_{t+1}R_{t+1})] \] (92)

- Observations:
  - by (87b), next period’s capital return determined by $R_{t+1} = \theta_{t+1}f'(k_{t+1})$
  - uncertainty in $R_{t+1}$ completely due uncertainty about shock which has distribution $\mu$ independent of any other realizations at time $t$
  - this permits (92) to be written as:
    \[ u'(w_t - k_{t+1}) = \beta \mathbb{E}_\mu [R(k_{t+1}, \cdot)u'(k_{t+1}R(k_{t+1}, \cdot))] \] (93)
    \[ = \beta \int_\Theta R(k_{t+1}, \theta)u'(k_{t+1}R(k_{t+1}, \theta))\mu(d\theta). \]

- Consumption of both generations in $t$ satisfies:
  \[ c^y_t = w_t - k_{t+1} \] (94a)
  \[ c^o_t = R_t k_t. \] (94b)
4.1.4 Equilibrium

- Stochastic OLG economy is summarized by the list \( E_{SOLG} = \langle u, \beta, f, \mu \rangle \)

- Following definition of equilibrium is straightforward generalization of deterministic case.

**Definition 4.1** Given \( k_0 > 0 \), an equilibrium of \( E_{SOLG} \) consists of adapted stochastic processes of prices \((w_t^e, R_t^e)_{t \geq 0}\) and an allocation \((k_{t+1}^e, c_t^y, c_t^o)_{t \geq 0}\) satisfying (87), (93), and (94) for all \( t \geq 0 \).

- Questions as in the deterministic case:
  - existence of equilibrium?
  - uniqueness of equilibrium?
  - dynamic behavior of equilibrium?

- To answer them, will again derive recursive structure of equilibrium.
4.1.5 Recursive structure of equilibrium

- Following ideas exactly analogous to deterministic case studied in Section 1.6.5

- Given $k > 0$ and $\theta \in \Theta$, define for each $0 < k_+ < W(k, \theta)$ the function

$$H(k_+; k, \theta) := u'(W(k, \theta) - k_+) - \beta E_\mu[\mathcal{R}(k_+, \cdot) u'(k_+ \mathcal{R}(k_+, \cdot))].$$ (95)

- Equilibrium process $(k^e_{t+1})_{t \geq 0}$ solves $H(k^e_{t+1}; k^e_t, \theta_t) = 0$ for all $t \geq 0$ and determines all other equilibrium variables

- Following auxiliary result can be proved exactly as in the deterministic case:

\textbf{Lemma 4.2} Under Assumptions 1.2 and 1.3, the following holds:

(i) The function $H(\cdot; k, \theta)$ defined in (95) has at least one zero for all $k > 0$ and $\theta \in \Theta$.

(ii) If, in addition either (a) or (b) of Assumption hold, this zero is unique.
Lemma 4.2 allows us to state the following main result:

**Proposition 4.1** Under Assumptions 1.2 and 1.3, the following holds for all $k_0 > 0$:

1. Economy $E_{\text{SOLG}}$ has at least one equilibrium.
2. If, in addition, either (a) or (b) of Assumption 1.4 hold, this equilibrium is unique.

- Observations:
  - additional restrictions ensure existence of a map $K : \mathbb{R}_+ \times \Theta \rightarrow \mathbb{R}_+$ which determines the unique solution $k_+ = K(k, \theta)$ to (34) for each $k > 0$ and $\theta \in \Theta$.
  - by the implicit function theorem, $K$ is $C^1$, strictly increasing, and satisfies
    \[
    0 < K(k, \theta) < \mathcal{W}(k, \theta) < f(k, \theta). \tag{96}
    \]
  - unique equilibrium process $(k_{t+1}^e)_{t \geq 0}$ determined recursively by
    \[
    k_{t+1}^e = K(k_t^e, \theta_t). \tag{97}
    \]

- To study equilibrium dynamics, need some basic concepts from stochastic dynamical systems theory.
5 Stochastic dynamical systems in discrete time

5.1 Stochastic dynamical systems and stability

• For more details, the reader is again referred to Stachurski (2009).

• Assume that endogenous state dynamics take the form

\[ x_{t+1} = F(x_t, \theta_t) \]  (98)

where we now restrict attention to case where \( X = \mathbb{R}_+ \)

• Also assume that exogenous process is i.i.d. with distribution \( \mu_\theta \) and values in \( \Theta = [\theta_{\text{min}}, \theta_{\text{max}}] \subset \mathbb{R}_{++} \)

• In the deterministic case, the state \( x_t \) in period \( t \) is a real number

• In the stochastic case, the state \( x_t \) in period \( t \) is a random variable which is completely described by its distribution \( \mu_t : \mathcal{B}(X) \rightarrow [0, 1] \)

• Thus, a steady state in the stochastic case is a distribution \( \bar{\mu} \) (or a random variable \( \bar{x} \) which has this distribution) which remains invariant under (98).

• Thus, to compute a stochastic steady state of (98), we need to derive how the sequence of distributions \((\mu_t)_{t\geq0}\) of the random variables \((x_t)_{t\geq0}\) evolve over time
5.2 Markov operator

- Suppose $x_0$ has distribution $\mu_0$, what is the distribution $\mu_t$ of $x_t$ for any $t \geq 1$?

- As in Section 3.2.2, note that (98) defines a transition probability, i.e., a mapping $Q : X \times \mathcal{B}(X) \rightarrow [0, 1]$ such that $Q(x, A)$ is the probability that $x_{t+1} \in A$ given that $x_t = x$.

- For all $x \in X$ and $A \in \mathcal{B}(X)$, $Q$ can explicitly be constructed as
  \[
  Q(x, A) = \mu_{\theta}\{\theta \in \Theta | F(x, \theta) \in A\} \tag{99}
  \]

- The distributions $(\mu_t)_{t \in \mathbb{T}}$ can then be computed recursively for $t \geq 1$ as
  \[
  \mu_t(B) = \int_X Q(x, B) \mu_{t-1}(dx). \tag{100}
  \]
  for each $B \in \mathcal{B}(X)$.

- Let $\mathcal{M}(X)$ denote the class of probability measures on $\mathcal{B}(X)$.

- Then, can define an operator $T : \mathcal{M}(X) \rightarrow \mathcal{M}(X)$ which associates with any $\mu \in \mathcal{M}(X)$ the new probability measure $T \mu$ defined for each $B \in \mathcal{B}(X)$ as
  \[
  T \mu(B) = \int_X Q(x, B) \mu(dx). \tag{101}
  \]
5.3 Stochastic steady states

- The concept of an invariant distribution is now straightforward:

**Definition 5.1** An steady state of the stochastic dynamical system \((98)\) is a probability distribution \(\bar{\mu} \in \mathcal{M}(\mathbb{X})\) which is a fixed point of \(T\), i.e., \(T\bar{\mu} = \bar{\mu}\).

- The stochastic analog of a steady state is therefore an invariant probability distribution

- Large literature which studies existence of invariant distributions for Markov operators

- Notion of stability requires \(\lim_{t \to \infty} T^t\mu_0 = \bar{\mu}\) where the limiting operation requires a suitable notion of convergence of measures (most results on stability use the concept of weak convergence, see Stokey & Lucas (1989)).

- Very general conditions for existence/uniqueness/stability of invariant distributions if \(F\) resp. \(T\) has certain monotonicity properties in Kamihigashi & Stachurski (2014)

- There is also a theory of *Random Dynamical Systems* due to Arnold (1998) which defines the concept of a random fixed point.

- See Schenk-Hoppé & Schmalfuss (2001) for an economic application of this theory and how it relates to the previous concepts
5.4 Equilibrium dynamics in the stochastic OLG model

5.4.1 Stable sets

- Consider existence of stochastic steady states/invariant distributions of economy $\mathcal{E}_{\text{SOLG}}$
- Existence of stochastic steady states follows from the existence of stable sets:

**Definition 5.2** A stable set of (97) is an interval $[k_{\text{min}}, k_{\text{max}}] \subset \mathbb{R}^{++}$ such that:

(i) $\mathcal{K}(k_{\text{min}}, \theta_{\text{min}}) = k_{\text{min}}$
(ii) $\mathcal{K}(k_{\text{max}}, \theta_{\text{max}}) = k_{\text{max}}$
(iii) $\mathcal{K}(k, \theta_{\text{min}}) < k < \mathcal{K}(k, \theta_{\text{max}})$ for all $k \in [k_{\text{min}}, k_{\text{max}}]$

- Existence of a stable set non-trivial steady state $\bar{k} > 0$ not guaranteed, fails if

$$\mathcal{K}(k, \theta_{\text{max}}) < k$$

for all $k > 0$ (impoverishment).

- Sufficient condition to exclude this and ensure existence is

$$\lim_{k \searrow 0} \mathcal{K}'(k, \theta_{\text{min}}) > 1.$$
5.4.2 Existence of a stochastic steady state

• Following existence result due to Wang (1993):

**Proposition 5.1 (Wang (1993))** *If the equilibrium map $K$ from (97) satisfies condition (103), there exists at least one stochastic steady state/invariant probability distribution.***

• Uniqueness of a stable sets not guaranteed, same multiplicity problem as in the deterministic case.

• Uniqueness obtains, however, if for all $\theta \in \Theta$, $K(\cdot, \theta)$ has a unique fixed point.

• This is a special case of the more general concept of a *stable fixed point configuration* (cf. Brock & Mirman (1972)). Essentially, this requires that the largest fixed point of $K(\cdot, \theta_{\text{min}})$ be smaller than the smallest fixed point of $K(\cdot, \theta_{\text{max}})$ (cf. the illustrations provided in class).

• Much more general existence results on stochastic steady states that also hold for a much larger class of OLG economies can be found, e.g., in Morand & Reffett (2007) and McGovern, Morand & Reffett (2013).
6 Stochastic decision problems with infinite horizon

6.1 A prototype decision problem

- Consider the problem as in Section 2.2 with $\mathbb{T} := \{0, 1, 2, \ldots\}$ but now with uncertainty.
- In particular, we now:
  - abstract from loans by requiring $s_t \geq 0$.
  - include the consumer’s labor-leisure choice $h_t \in [0, \bar{h}]$ which determines labor supply.
- Remainder normalizes maximum labor to $\bar{h} = 1$. 

6.1.1 Decision setup

• Given variables:
  ○ adapted stochastic process of wages \( w^\infty = (w_t)_{t \in \mathbb{T}} \)
  ○ adapted stochastic process of capital returns \( R^\infty = (R_t)_{t \in \mathbb{T}} \)
  ○ initial capital \( \bar{s}_{-1} \geq 0 \)

• Decision variables:
  ○ consumption plan: adapted stochastic process \( (c_t)_{t \in \mathbb{T}} \geq 0, c_t \geq 0 \ \forall t \)
  ○ investment plan: adapted stochastic process \( (s_t)_{t \in \mathbb{T}}, s_t \geq 0 \ \forall t \)
  ○ labor supply plan: adapted stochastic process \( (h_t)_{t \in \mathbb{T}}, 0 \leq h_t \leq \bar{h} \ \forall t \)
6.1.2 Intertemporal budget set

- Feasible plans must satisfy period budget equation

\[ c_t + s_t \leq w_t h_t + R_t s_{t-1} \]  \hspace{2cm} (104)

for all \( t \in \mathbb{T} \) where \( s_{-1} = \bar{s}_{-1} \)

- Feasible plans are defined by budget set:

\[ \mathbb{B}(w^\infty, R^\infty, \bar{s}_{-1}) = \left\{ (c_t, h_t, s_t)_{t \in \mathbb{T}} \mid c_t \geq 0, 0 \leq h_t \leq 1, s_t \geq 0, \text{(104) holds for all } t \in \mathbb{T} \right\} \]  \hspace{2cm} (105)
6.1.3 Preferences and decision problem

- Utility in period $t$ now depends on consumption $c_t \geq 0$ and leisure $0 \leq h_t \leq 1$ and given by utility function

$$u : \mathbb{R}_+ \times [0,] \longrightarrow \mathbb{R}, \ (c, h) \mapsto u(c, h) \tag{106}$$

- Preferences over consumption-labor processes $(c_t, h_t)_{t \in \mathbb{T}}$ represented by utility function

$$U((c_t, h_t)_{t \in \mathbb{T}}) := \mathbb{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t u(c_t, h_t) \right], \ 0 < \beta < 1. \tag{107}$$

- Decision problem:

$$\max \left\{ U((c_t, h_t)_{t \in \mathbb{T}}) \left| (c_t, h_t, s_t)_{t \in \mathbb{T}} \in \mathbb{B}(e^\infty, R^\infty, \bar{s}_1) \right. \right\}. \tag{108}$$

**Assumption 6.1** The utility function $u$ in (106) is continuous, strictly concave and $C^2$ on the interior of its domain with partial derivatives satisfying

$$\frac{\partial^2 u}{\partial c^2} < 0 < \frac{\partial u}{\partial c} \quad \text{and} \quad \lim_{c \searrow 0} \frac{\partial u}{\partial c}(c, h) = \infty \tag{109a}$$

$$\frac{\partial^2 u}{\partial h^2} < 0 < -\frac{\partial u}{\partial h} \quad \text{and} \quad \lim_{h \nearrow 1} \frac{\partial u}{\partial h}(c, h) = -\infty. \tag{109b}$$
6.2 Solving the decision problem

- Following derivations impose Assumption 6.1
- Then, any solution to (108) will be interior, i.e., $c_t^* > 0$ and $0 < h_t^* < 1$ due to (109)
- Can again use a variational argument to obtain following equations which characterize solution
- For each $t \geq 0$ and conditional on $\mathcal{F}_t$, solution to (108) must satisfy the intratemporal optimality condition
  \[
  - \frac{\partial h u(c_t, h_t)}{\partial c u(c_t, h_t)} = w_t
  \]  
  and the intertemporal optimality condition (Euler equation)
  \[
  \mathbb{E}_t \left[ R_{t+1} \frac{\beta \partial c u(c_{t+1}, h_{t+1})}{\partial c u(c_t, h_t)} \right] = 1.
  \]
- Further, for all $t \geq 0$, the budget equality
  \[
  c_t + s_t = w_t h_t + R_t s_{t-1}
  \]
holds and the stochastic transversality condition (STVC)

\[
\lim_{T \to \infty} \mathbb{E}_0 \left[ s_T \prod_{t=0}^{T} R_t^{-1} \right] = 0. \tag{113}
\]

• Remarks:

- that any process \((c_t^*, s_t^*, h_t^*)_{t \in T} \in \mathbb{B}(e^{\infty}, R^{\infty}, \tilde{s}_{-1})\) satisfying (110), (111), (112) for all \(t \in T\) as well as (113) is indeed a solution to (108) can be proved along the lines of the proof of Proposition 2.3 done in class (exploiting the law of iterated expectations!).

- one can also show by using the same arguments as in the proof of Proposition 2.3 that the solution to (108) is \(\mathbb{P}\)-a.s. unique.

- we could also - somewhat mechanically - have used a Lagrangian approach to obtain the previous conditions, but it is not quite clear how derivatives conditional on \(\mathcal{F}_t\) should be interpreted.
6.3 An equilibrium framework: The RBC model

- Consider a stochastic version of the neoclassical growth model from Section 2.4 with endogenous labor supply:
  
  - production side modified to incorporate random production shocks
  - consumer side modified to include labor-leisure choice, decision problem solved under uncertainty as in Section 6.1
  - unless stated otherwise, all other assumptions remain the same as in Section 2.4

- We continue to denote equilibrium variables as \((X_t)_{t \geq 0}\) but these are now adapted stochastic processes rather than just sequences.

- All equalities and inequalities involving random variables are assumed to hold \(\mathbb{P}\)-almost surely without explicit notice.
6.3.1 Consumer side

- As in deterministic case, $N$ identical consumers who each
  - plan over infinitely many future periods $\mathbb{T} = \{0, 1, 2, \ldots\}$
  - consume $c_t$ and invest $s_t$ in period $t$
  - supply $h_t$ units of labor in period $t$, now determined endogenously
  - capital earns return $R_t$, labor the wage $w_t$ in $t$

- Decision problem exactly as in Section 6.1

- As consumers are identical, so are the decisions they take!

- At the aggregate level, factor supply in period $t \geq 0$ given by
  \begin{align*}
  L_t &= Nh_t \\
  K_t &= Ns_{t-1}
  \end{align*}
  \hspace{1cm} (114) (115)

  and capital per capita $k_t := K_t/N$ evolves as
  \begin{equation}
  k_t = s_{t-1}, \quad t \geq 1. \hspace{1cm} (116)
  \end{equation}
• Optimal decision satisfies the conditions:

\[ c_t + k_{t+1} = w_t h_t + R_t k_t \]  \hspace{1cm} (117a)

\[-\frac{\partial h u(c_t, h_t)}{\partial c u(c_t, h_t)} = w_t \]  \hspace{1cm} (117b)

\[ \mathbb{E}_t \left[ R_{t+1} \frac{\beta \partial u(c_{t+1}, h_{t+1})}{\partial u(c_t, h_t)} \right] = 1 \]  \hspace{1cm} (117c)

and the stochastic transversality condition (STVC)

\[ \lim_{T \to \infty} \mathbb{E}_0 \left[ k_{T+1} \prod_{t=0}^{T} R_t^{-1} \right] = 0. \]  \hspace{1cm} (118)
6.3.2 Production side

- Suppose that (net) production in period \( t \) is subject to multiplicative shock \( \theta_t \in \Theta \subset \mathbb{R}_{++} \) such that total output (including non-depreciated capital) is given by:

\[
Y_t = e^{\theta_t} F(K_t, L_t) + (1 - \delta)K_t = N \left[ e^{\theta_t} h_t f(k_t/h_t) + (1 - \delta)k_t \right]
\]  

(119)

- Continue to assume linear homogeneity of \( F \) and impose Assumption \[1.3\] on \( f \).

- Using (114) and (115) and linear homogeneity, per capita output \( y_t := Y_t/N \) given by

\[
y_t = e^{\theta_t} F(k_t, h_t) + (1 - \delta)k_t = e^{\theta_t} h_t f(k_t/h_t) + (1 - \delta)k_t
\]

(120)

- Production shocks \( (\theta_t)_{t \geq 0} \) follow an AR(1)-process of the form

\[
\theta_t = \rho \theta_{t-1} + \varepsilon_t
\]

(121)

where \( 0 \leq \rho < 1 \) and \( (\varepsilon_t)_{t \geq 0} \) consist of i.i.d. random variables with distribution \( \mu_\varepsilon \)

- Thus, we can choose the underlying probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and construct transition probability \( Q \) induced by (121) exactly as outlined in Section 3.2.2

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• In each period $t$, the firm takes the current shock in period $t$ as given and decides on demand for capital and labor.

• The first order conditions then determine equilibrium factor prices as:

$$w_t = \mathcal{W}(k_t, h_t, \theta_t) := e^{\theta_t} \partial_h F(k_t, h_t)$$
$$= e^{\theta_t} [f(k_t/h_t) - k_t/h_t f'(k_t/h_t)]$$

$$R_t = \mathcal{R}(k_t, h_t, \theta_t) := e^{\theta_t} \partial_k F(k_t, h_t) + (1 - \delta)$$
$$= e^{\theta_t} f'(k_t/h_t) + 1 - \delta$$

• Remark:

  ° in the deterministic case, non-depreciated capital was included in $F$ resp. $f$ which had the interpretation of a gross production function

  ° here, we interpret $F$ resp. $f$ as a net production function and must, therefore, explicitly keep track of non-depreciated capital

  ° the reason is that only net production output is affected by the shock.
6.4 Equilibrium

- Economy is summarized by the list \( \mathcal{E}_{RBC} = \langle u, \beta, N, f, Q \rangle \) plus initial conditions \( k_0 > 0 \) and \( \theta_0 \in \Theta \)

**Definition 6.1** Given \( k_0 > 0 \) and \( \theta_0 \in \Theta \), an equilibrium of \( \mathcal{E}_{RBC} \) is an allocation \( (c^e_t, h^e_t, k^e_{t+1})_{t \geq 0} \) and a price sequence \( (w^e_t, R^e_t)_{t \geq 0} \) which satisfy (117) and (122) for all \( t \geq 0 \) and (118).

- Can again use an equivalent planning problem to determine the (unique) equilibrium allocation

- Equilibrium prices then follow directly from (122) for all \( t \geq 0 \).
6.5 A stochastic planning problem

- Consider a benevolent social planner who maximizes consumer utility by choosing a feasible allocation.

**Definition 6.2** Given $k_0 > 0$ and $\theta_0 \in \Theta$, a feasible allocation is an adapted stochastic process $(c_t, h_t, k_{t+1})_{t \geq 0}$ which satisfies $c_t \geq 0$, $0 \leq h_t \leq 1$, $k_{t+1} \geq 0$ for all $t \geq 0$ as well as the resource constraint

$$k_{t+1} + c_t \leq e^{\theta_t} F(k_t, h_t) + (1 - \delta)k_t.$$ (123)

The set of feasible allocations is denoted $\mathbb{A}(k_0, \theta_0)$.

- The planning problem reads:

$$\max_{(c_t, h_t, k_{t+1})_{t \in \mathbb{T}}} \left\{ U((c_t, h_t)_{t \in \mathbb{T}}) \left| (c_t, h_t, k_{t+1})_{t \in \mathbb{T}} \in \mathbb{A}(k_0, \theta_0) \right. \right\}$$ (124)

- As in the deterministic case, can compute the equations that characterize the solution to (124).

- Can show that these coincide with the equilibrium equations derived above.

- Thus, the solution to (68) also constitutes an equilibrium allocation!
6.6 Solving the stochastic planning problem by recursive methods

6.6.1 The Bellman equation

- Motivation for the following approach is analogous to the deterministic case
- Basic idea: Exploit the recursive structure of SPP
- Assume that $f$ satisfies Assumption 1.3 and $u$ Assumption 6.1 and $0 < \beta < 1$
- For brevity, set

$$M(k, h, \theta) := e^\theta F(k, h) + (1 - \delta)k$$  \hspace{1cm} (125)

- In the present stochastic setup, the Bellmann equation reads:

$$V(k, \theta) = \max_{k_+ \geq 0, 0 \leq h \leq 1} \left\{ u(M(k, h, \theta) - k_+, h) + \beta \int_{\Theta} V(k_+, \theta_+)Q(\theta, d\theta_+) \mid k_+ \leq M(k, h, \theta) \right\}$$  \hspace{1cm} (126)
6.6.2 Policy function

- Having computed the value function $V$, suppose the maximizing solution $(k^*_+, h^*)$ in (126) is well-defined and unique for each $(k, \theta) \in \mathbb{R}_{++} \times \Theta$

- Define the policy function $g = (g_k, g_h): \mathbb{R}_{++} \times \Theta \rightarrow \mathbb{R}_+ \times [0, 1]$

$$g(k, \theta) = \arg \max_{k_+ \geq 0, 0 \leq h \leq 1} \left\{ u(M(k, h, \theta) - k_+, h) + \beta \int_{\Theta} V(k_+, \theta_+) Q(\theta, d\theta_+) | k_+ \leq M(k, h, \theta) \right\}$$

Lemma 6.1 Let $V$ be the unique solution to (126) and $g = (g_k, g_h)$ be defined as above. Then, for each $(k_0, z_0)$ the sequence $\{c^*_t, h^*_t, k^*_t+1\}_{t \geq 0}$ defined recursively as $k^*_0 = k_0$,

$$k^*_{t+1} = g_k(k^*_t, \theta_t)$$
$$h^*_t = g_h(k^*_t, \theta_t)$$
$$c^*_t = M(k^*_t, h^*_t, \theta_t) - k^*_{t+1}$$

for all $t \geq 0$ is a solution to (124).
6.7 Equilibrium dynamics in the RBC model

• Consequences of previous results:
  
  ○ dynamics completely described by the endogenous state variable \( \{k_t^*\}_{t \geq 0} \) and the exogenous process \( \{\theta_t\}_{t \geq 0} \)
  
  ○ analogously to stochastic OLG model, can analyze dynamics, existence of invariant distributions, etc.

  ○ in general, mapping \( g_k(\cdot; \theta) \) has a unique steady state \( \bar{k}_\theta \) for all \( \theta \in \Theta \)
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