Mathematical Methods, Part 1: Applied Intertemporal Optimization

Part I

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Lecturer: Marten Hillebrand
Teaching assistant: Van Hoang Khieu

Part I
Deterministic Models in Discrete Time

1 Decision problems with finite time horizon

1.1 A prototype consumption investment problem

1.1.1 Setup

- Consider single consumer/household:
  - takes decisions in *initial period* $t = 0$
  - life/planning horizon ends in *terminal period* $t = T > 0$
  - *planing horizon* is $\mathbb{T} := \{0, 1, \ldots, T\}$
- Single consumption good ('numeraire'), all quantities denominated in consumption units
- In each period $t \in \mathbb{T}$:
  - consumer earns exogenous non-capital income $e_t \geq 0$
  - consumes $c_t \geq 0$, invests/borrows capital s_t
  - one unit invested in $t - 1$ earns gross return $R_t > 0$ in t
- Thus, to make problem more interesting, allow for unbounded loans
- But: require $s_T \geq 0$, i.e., no loans in terminal period

1.1.2 Intertemporal budget set

- Given quantities in decision:
  - non-capital incomes $e^T := (e_0, \ldots, e_T)$
  - returns $R^T := (R_0, \ldots, R_T)$
  - initial capital $\bar{s}_{-1}$
- Decision variables:
  - consumption plan $(c_t)_{t \in \mathbb{T}}$
  - investment plan $(s_t)_{t \in \mathbb{T}}$
- Sequential budget constraint for all $t \in \mathbb{T}$:

$$c_t + s_t \leq e_t + R_t s_{t-1} \tag{1}$$

  where $s_{-1} \equiv \bar{s}_{-1}$

- Incomes, returns, and initial capital determine budget set:

$$\mathbb{B}(e^T, R^T, \bar{s}_{-1}) := \left\{(c_t, s_t)_{t \in \mathbb{T}} | c_t \geq 0, (1) \text{ holds for all } t \in \mathbb{T}, s_T \geq 0\right\} \tag{2}$$

- Investment sequence $(s_t)_{t\in\mathbb{T}}$ determines future wealth levels

$$W_t := e_t + R_t s_{t-1}, \quad t > 0. \tag{3}$$

- Define discounted future lifetime income

$$E_t := \frac{e_{t+1}}{R_{t+1}} + \ldots + \frac{e_T}{R_{t+1}\cdots R_T} = \sum_{n=t+1}^{T} e_n \prod_{m=t+1}^{n} R_m^{-1} \tag{4}$$

for $t \in \mathbb{T}\backslash\{T\}$ and $E_T := 0$.

Lemma 1.1 *The budget set defined in (2) satisfies the following:*

(i) Any $(c_t, s_t)_{t\in\mathbb{T}} \in \mathbb{B}(e^T, R^T, \bar{s}_{-1})$ *satisfies*

$$s_t \geq \underline{s}_t := -E_t \tag{5}$$

for all $t \in \mathbb{T}$ *while the wealth levels defined in (3) satisfy* $W_t \geq -E_t$.

(ii) $\mathbb{B}(e^T, R^T, \bar{s}_{-1})$ *is non-degenerate (contains more than one element) iff*

$$\bar{s}_{-1} > -\frac{e_0 + E_0}{R_0}. \tag{6}$$

(iii) $\mathbb{B}(e^T, R^T, \bar{s}_{-1})$ *is compact and convex* $\forall (e^T, R^T, \bar{s}_{-1}) \in \mathbb{R}_{+}^{T+1} \times \mathbb{R}_{++}^{T+1} \times \mathbb{R}$ *satisfying (6).*

- Remainder assumes that solvency condition (6) holds.

1.1.3 Preferences and decision problem

- Consumer has time-additive utility function $U : \mathbb{R}_+^{T+1} \longrightarrow \mathbb{R}$,

$$U((c_t)_{t\in\mathbb{T}}) = \sum_{t=0}^{T} \beta^t u(c_t), \quad \beta > 0. \tag{7}$$

- Decision problem reads:

$$\max\Big\{U((c_t)_{t\in\mathbb{T}})|(c_t, s_t)_{t\in\mathbb{T}} \in \mathbb{B}(e^T, R^T, \bar{s}_{-1})\Big\}. \tag{8}$$

- **Show:** Following restriction on u sufficient for (8) to have a unique solution:

  Assumption 1.1 *The period utility function $u : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is continuous, strictly increasing, and strictly concave.*

- Following stronger restriction will be convenient more:

  Assumption 1.2 *The period utility function $u : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is continuous and C^2 on $\mathbb{R}_{++}$ with derivatives satisfying $u'' < 0 < u'$ and the Inada condition $\lim_{c\searrow 0} u'(c) = \infty$.*

1.2 Solving the problem by Lagrangian methods

- Adopt a standard Lagrangian approach to solve (8)
- Let u satisfy stronger Assumption 1.2 and let solvency condition (6) hold
- Define the Lagrangian function

$$\mathscr{L}((c_t, s_t, \mu_t, \lambda_t)_{t\in\mathbb{T}}) := \sum_{t\in\mathbb{T}} \left[\beta^t u(c_t) + \mu_t c_t + \lambda_t(e_t + R_t s_{t-1} - s_t - c_t)\right]$$

- Standard arguments imply that $(c_t^*, s_t^*)_{t\in\mathbb{T}}$ solves (8) if there exist non-negative Lagrangian multipliers $(\mu_t^*, \lambda_t^*)_{t\in\mathbb{T}}$ such that $(c_t^*, s_t^*, \mu_t^*, \lambda_t^*)_{t\in\mathbb{T}}$ solves the first order conditions (FOCs):

$$\frac{\partial \mathscr{L}}{\partial c_t}((c_t, s_t, \mu_t, \lambda_t)_{t\in\mathbb{T}}) = \beta^t u'(c_t) + \mu_t - \lambda_t = 0 \quad \forall t \in \mathbb{T} \tag{9}$$

$$\frac{\partial \mathscr{L}}{\partial s_t}((c_t, s_t, \mu_t, \lambda_t)_{t\in\mathbb{T}}) = -\lambda_t + R_{t+1}\lambda_{t+1} = 0 \quad \forall t \in \mathbb{T}\backslash\{T\} \tag{10}$$

the complementary slackness conditions (CSCs):

$$\mu_t c_t = \lambda_t(e_t + R_t s_{t-1} - s_t - c_t) = 0 \tag{11}$$

and (1) and $c_t \geq 0$ for all $t \in \mathbb{T}$ where $s_{-1} = \bar{s}_{-1}$ and $s_T = 0$.

- Observations:
  - boundary behavior of u' from Assumption 1.2 excludes $c_t^* = 0$
  - thus, $\mu_t^* = 0$ for all $t \in \mathbb{T}$ by (11)
  - then, by (9) and (10), $\lambda_t^* > 0$
  - hence, (1) is binding for all $t \in \mathbb{T}$ by (11).
- These results give:

  Proposition 1.1 *Let Assumption 1.2 and the solvency condition (6) be satisfied. Then, any list* $(c_t^*, s_t^*)_{t\in\mathbb{T}}$ *which solves the conditions*

$$\beta u'(c_{t+1})R_{t+1} = u'(c_t) \qquad \forall t \in \mathbb{T}\backslash\{T\} \tag{12a}$$

$$c_t + s_t = e_t + R_t s_{t-1} \quad \forall t \in \mathbb{T} \tag{12b}$$

$$s_T = 0 \tag{12c}$$

  is a solution to (8).

- Interpretation of (12a) ('MRS' = 'marginal rate of substitution'):

$$\underbrace{\frac{\beta u'(c_{t+1})}{u'(c_t)}}_{\text{intertemporal MRS}} = \underbrace{\frac{1}{R_{t+1}}}_{\text{intertemporal price ratio}}$$

1.3 Solving the problem by recursive methods

1.3.1 The three-period case

- Impose [weaker] Assumption 1.1 on u, suppose first $T = 2$
- Consider future decision problem in $t = T - 1 = 1$ given arbitrary $W_1 = e_1 + R_1 s_0 \geq -E_1$:

$$\max_{c_1, s_1} \Big\{ u(c_1) + \beta u(e_2 + s_1 R_2) | c_1 \geq 0, c_1 + s_1 \leq W_1, s_1 \geq -E_1 \Big\}$$

- For each $W \geq -E_1$, define the *value function* [**why well-defined?**]

$$V_1(W) := \max_{c_1, s_1} \Big\{ u(c_1) + \beta u(e_2 + s_1 R_2) | c_1 \geq 0, c_1 + s_1 \leq W, s_1 \geq -E_1 \Big\}.$$

- **Principle of Optimality** states that

$$\begin{aligned} & \max_{(c_t, s_t)_{t \in \mathbb{T}}} \Big\{ U((c_t)_{t \in \mathbb{T}}) | (c_t, s_t)_{t \in \mathbb{T}} \in \mathbb{B}(e^T, R^T, \bar{s}_{-1}) \Big\} \\ = \quad & \max_{c_0, s_0} \Big\{ u(c_0) + \beta V_1(e_1 + s_0 R_1) \,|\, c_0 \geq 0, s_0 \geq -E_0, c_0 + s_0 \leq e_0 + R_0 \bar{s}_{-1} \Big\}. \end{aligned} \tag{13}$$

- Knowing V_1, obtain optimal decision (c_0^*, s_0^*) for $t = 0$ by solving one-stage problem (13)!

1.3.2 The general multi-period case

- Straightforward to generalize previous approach.
- Define value functions $(V_t)_{t\in\mathbb{T}}$ recursively by setting $V_T = u$ and, for all $W \geq -E_t$:

$$V_t(W) = \max_{c,s}\Big\{u(c) + \beta V_{t+1}(e_{t+1} + sR_{t+1}) \,|\, c \geq 0, s \geq -E_t, s + c \leq W\Big\} \tag{14}$$

- **Prove**: Under Assumption 1.1, each function V_t, $t \in \mathbb{T}$ is well-defined and continuous, strictly increasing, and strictly concave.
- Obtain the optimal decision in $t = 0$ as:

$$(c_0^*, s_0^*) = \arg\max_{c,s}\Big\{u(c) + \beta V_1(e_1 + sR_1) \,|\, c \geq 0, s \geq -E_0, c + s \leq e_0 + R_0\bar{s}_{-1}\Big\}. \tag{15}$$

- Can recover optimal decision for all $t \in \mathbb{T}$ recursively by setting $W_t^* = e_t + s_{t-1}^* R_t$ and

$$(c_t^*, s_t^*) = \arg\max_{c,s}\Big\{u(c) + \beta V_{t+1}(e_{t+1} + sR_{t+1}) \,|\, c \geq 0, s \geq -E_t, c + s \leq W_t^*\Big\}. \tag{16}$$

- Remark: A rigorous proof of the principle of optimality in a related context can be found in Hillebrand (2008).

1.4 Characterizing the recursive solution by first order conditions

1.4.1 Differentiability of the value functions

- Restrict u by stronger Assumption 1.2
- In this case, any candidate solution to (8) satisfies $W_t > -E_t$ for all $t \in \mathbb{T}$
- Using simple induction and the *implicit function theorem* (cf. Mas-Colell, Whinston & Green (1995), Appendix M.E), one (**you!** :-)) can show that
  - each value function V_t, $t \in \mathbb{T}$ is C^1
  - the solution (c_t^*, s_t^*) to (14) is determined by C^1 functions

$$\begin{aligned} C_t :]-E_t, \infty[&\longrightarrow \mathbb{R}_{++}, & c_t^* &= C_t(W_t) \\ S_t :]-E_t, \infty[&\longrightarrow]-E_t, \infty[, & s_t^* &= S_t(W_t). \end{aligned}$$

- Using this in (14) and the budget constraint gives for all $t < T$ and $W > -E_t$:

$$V_t(W) = u(W - S_t(W)) + \beta V_{t+1}(e_{t+1} + R_{t+1} S_t(W)) \tag{17}$$

- Also note that the optimal solution $S_t(W)$ satisfies the FOC's [**why?**]

$$-u'(W - S_t(W)) + \beta R_{t+1} V'_{t+1}(e_{t+1} + R_{t+1} S_t(W)). \tag{18}$$

1.4.2 The envelope theorem

- Differentiating (17) using (18) gives for all $t < T$ and $W > -E_t$:

$$V_t'(W) = u'(W - S_t(W)) = u'(C_t(W)). \tag{19}$$

- This is nothing but a simple application of the *envelope theorem* (see Mas-Colell, Whinston & Green (1995), Appendix M.L)

- We now clam that for all t and $W > -E_t$:

$$-u'(W - S_t(W)) + \beta R_{t+1} u'(e_{t+1} + R_{t+1} S_t(W)). \tag{20}$$

- To see this, suppose first $t = T - 1$. Then, $V_{t+1}' = u'$ can be used in (18) to obtain (20).

- Second, suppose $t < T - 1$. Then, (19) gives $V_{t+1}'(W) = u'(C_{t+1}(W))$ for all $W > -E_{t+1}$. Using this in (18) also gives (20).

- Defining the optimal wealth levels $(W_t^*)_{t \in \mathbb{T}}$ recursively by (3) and setting $c_t^* := C_t(W_t^*)$, equation (20) can be written as

$$-u'(c_t^*) + \beta R_{t+1} u'(c_{t+1}^*). \tag{21}$$

which is precisely the optimality condition (12a).

1.5 Handling problems with unbounded utility

- Some popular utility functions including log-utility satisfy $\lim_{c \searrow 0} u(c) = -\infty$
- Problem:
  - u not defined for $c = 0$
  - U not continuous on budget set $\rightsquigarrow$ above's existence argument fails!
- Thus, these functions are excluded by Assumption 1.1
- But: Can handle problem as follows:
  - choose a (very small) lower bound $\underline{c} > 0$
  - add restriction $c_t \geq \underline{c}$ to budget set (2)
  - choosing $\underline{c} > 0$ small enough ensures that $c_t^* > \underline{c}$ for all $t \in \mathbb{T}$
- Remark: Previous modification changes lower bounds on savings to

$$\underline{s}_t \geq -\hat{E}_t := -\left[\frac{e_{t+1} - \underline{c}}{R_{t+1}} + \ldots + \frac{e_T - \underline{c}}{R_{t+1} \cdots R_T}\right] = -\sum_{n=t+1}^{T} (e_n - \underline{c}) \prod_{m=t+1}^{n} R_m^{-1} \tag{22}$$

- If (6) holds for $\underline{c} = 0$, will continue to hold for $\underline{c} > 0$ small!

1.6 An equilibrium framework: The OLG model

1.6.1 Population structure

- Growth model with overlapping generations (OLG) of consumers provides natural framework for intertemporal decision problems with finite horizon
- Consider simplest case with stationary population of two-period lived consumers:
  - new generation of $N \geq 0$ consumers born in each period $t \geq 0$
  - these consumers live for two periods, die at end of $t + 1$
  - generational index $j \in \{y, o\}$ identifies 'young' and 'old' generation in $t \geq 0$
- Young consumer in period t:
  - supplies one unit of labor
  - saves/invests s_t
  - consumes c_t^y
- Old consumer in period t:
  - supplies capital $k_t = s_{t-1}$
  - consumes c_t^o

1.6.2 Consumer behavior

- A young consumer in period $t \geq 0$
  - labor income $w_t > 0$ in t, no labor income in $t+1$
  - chooses savings $s_t \geq 0$ and lifetime consumption $(c_t^y, c_{t+1}^o) \geq 0$ subject to:

$$c_t^y = w_t - s_t \tag{23a}$$

$$c_{t+1}^o = R_{t+1} s_t \tag{23b}$$

  - lifetime utility function $U(c^y, c^o) := u(c^y) + \beta u(c^o)$, $\beta > 0$
  - decision problem:

$$\max_s \Big\{ u(w_t - s) + \beta u(sR_{t+1}) \mid 0 \leq s \leq w_t \Big\} \tag{24}$$

- Impose Assumption 1.2 on utility u.

- Optimal decision:
  - special case of decision problem ($T = 1$, $e_0 = w_t$, $e_1 = 0$)
  - unique solution determined by first order conditions

Lemma 1.2 *Let Assumption 1.2 be satisfied. Then, for each* $(w_t, R_{t+1}) \gg 0$*, there exists a unique solution* s_t *to (24) determined by*

$$u'(w_t - s_t) - R_{t+1}\,\beta u'(s_t R_{t+1}) = 0. \tag{25}$$

- Aggregate investment made at time t determines next periods's capital stock:

$$K_{t+1} = N s_t \tag{26}$$

- Defining per-capita capital stock $k_t := K_t/N$, (25) can be written as:

$$u'(w_t - k_{t+1}) - R_{t+1}\,\beta u'(k_{t+1} R_{t+1}) = 0. \tag{27}$$

- Consumption in t satisfies:

$$c_t^y = w_t - k_{t+1}. \tag{28}$$

- Old consumer in period $t \geq 0$ consumes his entire (capital) income:

$$c_t^o = R_t k_t. \tag{29}$$

1.6.3 Production side

- Representative firm produces output Y using labor and capital as inputs:

$$Y = F(K, L) \tag{30}$$

- Linear homogeneous technology F can be written as:

$$Y = Lf(K/L) \quad \text{where} \quad f(k) := F(k, 1) \tag{31}$$

- Remark: Interpret f as a *gross* production function that includes non-depreciated capital

Assumption 1.3 *The intensive form production function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is C^2 with derivatives satisfying $f'' < 0 < f'$, $\lim_{k\to 0} f'(k) = \infty$, and $\lim_{k\to\infty} f'(k) < 1$.*

- Given $(w_t, R_t) \gg 0$, firm chooses input demand to maximize profits:

$$\max_{K,L} \left\{ Lf\left(\frac{K}{L}\right) - R_t K - w_t L \mid (K, L) \in \mathbb{R}_+^2 \right\} \tag{32}$$

- FOCs of (32) determine equilibrium factor prices as function of capital intensity $k_t = \frac{K_t}{L_t}$:

$$w_t = \mathcal{W}(k_t) := f(k_t) - k_t f'(k_t) \tag{33a}$$
$$R_t = \mathcal{R}(k_t) := f'(k_t) \tag{33b}$$

1.6.4 Equilibrium

- OLG economy is summarized by the list $\mathscr{E}_{\text{OLG}} = \langle u, \beta, f \rangle$
- Following definition of equilibrium is standard

Definition 1.1 *Given $k_0 > 0$, an equilibrium of $\mathscr{E}_{\text{OLG}}$ consists of prices $(w_t^e, R_t^e)_{t \geq 0}$ and an allocation $(k_{t+1}^e, c_t^{y,e}, c_t^{o,e})_{t \geq 0}$ which satisfy equations (27), (28), (29), and (33) for all $t \geq 0$.*

- Questions:
  - existence of equilibrium?
  - uniqueness of equilibrium?
  - dynamic behavior of equilibrium?
- To answer them, will exploit recursive structure of equilibrium derived next

1.6.5 Recursive structure of equilibrium

- Given $k > 0$, define for each $0 < k_+ < \mathcal{W}(k)$ the function

$$H(k_+; k) := u'(\mathcal{W}(k) - k_+) - \beta\mathcal{R}(k_+)\, u'(k_+\mathcal{R}(k_+)). \tag{34}$$

- Equilibrium sequence $(k^e_{t+1})_{t\geq 0}$ satisfies $H(k^e_{t+1}; k^e_t) = 0$ for all $t \geq 0$ and determines all other equilibrium variables

- Uniqueness result derived below requires either of the following additional restriction:

  Assumption 1.4 *(a) The production function f satisfies $\frac{kf''(k)}{f'(k)} \geq -1$ for all $k > 0$.*

  (b) The utility function u satisfies $\frac{cu''(c)}{u'(c)} \geq -1$ for all $c > 0$.

- **Prove** the following auxiliary result:

  Lemma 1.3 *Under Assumptions 1.2 and 1.3, the following holds:*

  (i) The function $H(\cdot; k)$ defined in (34) has at least one zero for all $k > 0$.

  (ii) If, in addition either (a) or (b) of Assumption 1.4 hold, this zero is unique.

- Lemma 1.3 allows us to state the following main result:

Proposition 1.2 *Under Assumptions 1.2 and 1.3, the following holds for all $k_0 > 0$:*

(i) Economy $\mathscr{E}_{\mathrm{OLG}}$ has at least one equilibrium.

(ii) If, in addition, either (a) or (b) of Assumption 1.4 hold, this equilibrium is unique.

- Observations:
  - additional restrictions in Assumption 1.4 ensure existence of a map $\mathcal{K} : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ which determines the unique solution $k_+ = \mathcal{K}(k)$ to (34) for each $k > 0$
  - by the implicit function theorem, $\mathcal{K}$ is C^1, strictly increasing, and satisfies

$$0 < \mathcal{K}(k) < \mathcal{W}(k) < f(k) \tag{35}$$

  - unique equilibrium sequence $(k_{t+1}^e)_{t \geq 0}$ determined recursively by $k_0^e = k_0$ and

$$k_{t+1}^e = \mathcal{K}(k_t^e), \quad t \geq 0. \tag{36}$$

- To study equilibrium dynamics, need basic concepts from dynamical systems theory.

1.7 Dynamical systems in discrete time

1.7.1 State space, time-one map, orbits

- Let $\mathbb{X} \subset \mathbb{R}^N$ be
  - a non-empty (typically: open/closed, convex) set
  - endowed with Euclidean norm $\|\cdot\|$ and relative topology
- Let $\varphi : \mathbb{X} \longrightarrow \mathbb{X}$ be a function which maps $\mathbb{X}$ into itself
- $(\varphi, \mathbb{X})$ is a *deterministic dynamical system* with *time-one-map* φ and *state space* $\mathbb{X}$
- Each $x_0 \in \mathbb{X}$ induces a sequence $\{x_t\}_{t \geq 0}$ defined recursively as

$$x_{t+1} = \varphi(x_t) = \varphi(\varphi(x_{t-1})) = \varphi^2(x_{t-1}) = \varphi^{t+1}(x_0), \quad t > 0.$$

  where $\varphi^t = \underbrace{\varphi \circ \ldots \circ \varphi}_{t \text{ - times}} : \mathbb{X} \longrightarrow \mathbb{X}$, $t > 0$ and $\varphi^0 := \mathrm{id}_{\mathbb{X}}$
- For each $x_0 \in \mathbb{X}$, call

$$\Gamma(x_0) := (\varphi^t(x_0))_{t \geq 0}$$

  the *orbit* of x_0 (under φ)

1.7.2 Fixed points and stability

- Of particular interest: Points $\bar{x} \in \mathbb{X}$ which are limits of orbits and for which the dynamics become 'steady'

Definition 1.2 *A fixed point of dynamical system* $(\mathbb{X}, \varphi)$ *is a value* $\bar{x} \in \mathbb{X}$ *for which* $\bar{x} = \varphi(\bar{x})$*.*

- Orbit $\Gamma(\bar{x}) = (\bar{x}, \bar{x}, \ldots)$ is the constant sequence
- For which initial values $x_0 \in \mathbb{X}$ does $\Gamma(x_0)$ converge to fixed point $\bar{x}$, i.e.,

$$\lim_{t \to \infty} \varphi^t(x_0) = \bar{x}. \tag{37}$$

Definition 1.3 *A fixed point* $\bar{x} \in \mathbb{X}$ *of a dynamical system* $(\varphi, \mathbb{X})$ *is called*

(i) locally asymptotically stable, if there exists a neighborhood $U \subset \mathbb{X}$ *containing* $\bar{x}$ *such that* $\lim_{t \to \infty} \|\varphi^t(x) - \bar{x}\| = 0$ *for all* $x \in U$*.*

(ii) globally asymptotically stable if $\lim_{t \to \infty} \|\varphi^t(x) - \bar{x}\| = 0$ *for all* $x \in \mathbb{X}$*.*

(iii) unstable if it is not locally asymptotically stable.

- Clear:
  - global stability implies local stability
  - global stability requires φ to have a *unique* fixed point in $\mathbb{X}$
- Terminology:
  - unless stated otherwise, 'stable' taken to mean 'locally asymptotically stable'
  - fixed points synonymously referred to as 'steady states'

1.7.3 Analyzing the dynamics with phase diagrams

- done in class.

1.7.4 The Grobman-Hartman theorem

- Assume:
  - state space $\mathbb{X} \subset \mathbb{R}^N$ is an open set
  - time-one map φ is continuously differentiable (C^1)
- Next result:
  - sufficient criterion to infer local stability of $\bar{x}$ from Jacobian matrix $D\varphi(\bar{x})$
  - write $\varphi = (\varphi^{(1)}, \ldots, \varphi^{(N)})$ where $\varphi^{(n)} : \mathbb{X} \longrightarrow \mathbb{R}$, $n = 1, \ldots, N$

Lemma 1.4 (Grobman-Hartman Theorem) *Let φ be C^1 and $\bar{x} \in \mathbb{X}$ be a steady state of φ. Denote by $\lambda_n \in \mathbb{C}$, $n = 1, \ldots, N$ the Eigenvalues of the Jacobian matrix*

$$D\varphi(\bar{x}) = \begin{bmatrix} \frac{\partial \varphi^{(1)}}{\partial x^{(1)}}(\bar{x}) & \ldots & \frac{\partial \varphi^{(1)}}{\partial x^{(N)}}(\bar{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi^{(N)}}{\partial x^{(1)}}(\bar{x}) & \ldots & \frac{\partial \varphi^{(N)}}{\partial x^{(N)}}(\bar{x}) \end{bmatrix} \in \mathbb{R}^{N \times N}.$$

Then the following holds true:

(i) If $|\lambda_n| < 1$ for all $n \in \{1, \ldots, N\}$, then $\bar{x}$ is locally asymptotically stable.

(ii) If $|\lambda_n| > 1$ for at least one $n \in \{1, \ldots, N\}$, then $\bar{x}$ is unstable.

- In this course, we will mostly consider models where $N = 1$ in which case $\mathbb{X} \subset \mathbb{R}$ and the dynamics are one-dimensional

- A straightforward corollary to Lemma 1.4 is the following

  Corollary 1.1 *Let φ be C^1 on $\mathbb{X} \subset \mathbb{R}$ and $\bar{x} \in \mathbb{X}$ be an interior steady state of φ. Then, the following holds:*

  (i) If $|\varphi'(\bar{x})| < 1$, then $\bar{x}$ is locally asymptotically stable.

  (ii) If $|\varphi'(\bar{x})| > 1$, then $\bar{x}$ is unstable.

- Remarks:

  - a fixed point $\bar{x}$ for which $|\varphi'(\bar{x})| \neq 1$ is called *hyperbolic*
  - only stability properties of hyperbolic fixed points can be inferred from $\varphi'(\bar{x})$
  - in the non-hyperbolic case $|\varphi'(\bar{x})| = 1$, additional conditions must be checked.

1.8 Equilibrium dynamics in the OLG model

1.8.1 Existence and non-existence of steady states

- Know:
  - equilibrium dynamics in OLG model determined by map $\mathcal{K}$ defined in Section 1.6.5
  - $\mathcal{K}$ is C^1, strictly increasing, and bounded by $\mathcal{W}$
- But: Existence of a non-trivial steady state $\bar{k} > 0$ not guaranteed, may well have

$$\mathcal{K}(k) < k$$

  for all $k > 0$ ('impoverishment').

- Sufficient condition to exclude this and ensure existence of steady state $\bar{k} > 0$ is

$$\lim_{k \searrow 0} \mathcal{K}'(k) > 1.$$

- Monotonicity of $\mathcal{K}$:
  - implies monotonic convergence/divergence of all orbits
  - excludes cyclical behavior

1.8.2 Uniqueness and multiplicity of steady states

- Existence of a steady state does not imply uniqueness
- Examples discussed in class.
- For further discussion and explicit restrictions on fundamentals to obtain existence/uniqueness see Galor & Ryder (1989)

2 Decision problems with infinite time horizon

2.1 A prototype consumption investment problem

2.1.1 Decision setup

- Consider the same problem as in Section 1.2 but with $T = \infty$
- Planing horizon $\mathbb{T} := \{0, 1, 2, \ldots\}$ is now (countably) infinite
- Given variables:
  - sequence of non-capital incomes $e^\infty = (e_t)_{t\in\mathbb{T}} \geq 0$
  - sequence of capital returns $R^\infty = (R_t)_{t\in\mathbb{T}} \gg 0$
  - initial capital $\bar{s}_{-1}$ (to be restricted)
- Decision variables:
  - consumption plan $(c_t)_{t\in\mathbb{T}} \geq 0$
  - investment plan $(s_t)_{t\in\mathbb{T}}$

2.1.2 NPG-condition and intertemporal budget set

- As before, feasible plans satisfy budget equation

$$c_t + s_t \leq e_t + R_t s_{t-1} \tag{38}$$

  for all $t \in \mathbb{T}$ where $s_{-1} = \bar{s}_{-1}$

- For $t > 0$, define and interpret

$$q_t := R_1^{-1} \dots R_t^{-1} = \prod_{n=1}^{t} R_n^{-1} \tag{39}$$

  as price of time t consumption in units of time zero consumption [**why?**]

- Using (39), we also impose the No-Ponzi Game (NPG) condition

$$\lim_{t \to \infty} q_t s_t \geq 0. \tag{40}$$

- Interpretation of (40): All loans must ultimately be repaid!

- Feasible plans are defined by budget set:

$$\mathbb{B}(e^\infty, R^\infty, \bar{s}_{-1}) = \Big\{ (c_t, s_t)_{t \in \mathbb{T}} \big| \, c_t \geq 0, (38) \text{ holds for all } t \in \mathbb{T}, (40) \text{ holds} \Big\} \tag{41}$$

2.1.3 Preferences and decision problem

- Preferences over consumption plans $(c_t)_{t\in\mathbb{T}} \in \mathbb{R}_+^{\mathbb{T}}$ represented by utility function

$$U((c_t)_{t\in\mathbb{T}}) := \sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1. \tag{42}$$

- Decision problem:

$$\max\Big\{U((c_t)_{t\in\mathbb{T}})\Big|(c_t, s_t)_{t\in\mathbb{T}} \in \mathbb{B}(e^\infty, R^\infty, \bar{s}_{-1})\Big\}. \tag{43}$$

- Maintain Assumption 1.2 on utility and assume that $0 < \beta < 1$.
- Remark:
  - we will set aside problems with infinity by assuming that incomes $(e_t)_{t\in\mathbb{T}}$ and prices $(q_t)_{t\in\mathbb{T}}$ are 'well-behaved' such that $|U((c_t)_{t\in\mathbb{T}})| < \infty$ for all $(c_t, s_t)_{t\in\mathbb{T}} \in \mathbb{B}(e^\infty, R^\infty, \bar{s}_{-1})$
  - explicit conditions under wihch this holds can easily be formulated.

2.1.4 Lifetime budget constraint

- Remainder assumes that (38) holds with equality for all t (due to monotonic preferences)
- Use (38) (with equality) to recursively eliminate $(s_t)_{t\in\mathbb{T}}$ from decision
- Using (39), obtain for all $T \geq 1$:

$$q_T s_T = \sum_{t=0}^{T} q_t(e_t - c_t) + R_0 \bar{s}_{-1}. \tag{44}$$

- Taking the limit $T \to \infty$ and using (40) gives

$$\underbrace{\sum_{t=0}^{\infty} q_t c_t}_{\text{lifetime consumption expenditure}} \leq \underbrace{\sum_{t=0}^{\infty} q_t e_t}_{\text{lifetime income}} + \underbrace{R_0 \bar{s}_{-1}}_{\text{initial capital income}}. \tag{45}$$

- We will assume the solvency condition that lifetime wealth is positive and finite:

$$0 < M := \sum_{t=0}^{\infty} q_t e_t + R_0 \bar{s}_{-1} < \infty. \tag{46}$$

- Can now write (44) as

$$\sum_{t=0}^{\infty} q_t c_t \leq M. \tag{47}$$

to obtain lifetime budget set:

$$\tilde{\mathbb{B}}(q^{\infty}, M) := \Big\{(c_t)_{t\in\mathbb{T}} \Big| \, (46) \text{ holds } \wedge \, c_t \geq 0 \, \forall t \in \mathbb{T}\Big\}. \tag{48}$$

where $q^{\infty} := (q_t)_{t\in\mathbb{T}}$.

- Can state (43) in the following equivalent form:

$$\max\Big\{U((c_t)_{t\in\mathbb{T}}) \Big| (c_t)_{t\in\mathbb{T}} \in \tilde{\mathbb{B}}(q^{\infty}, M)\Big\}. \tag{49}$$

- Given a solution $(c_t^*)_{t\in\mathbb{T}}$ to (49), can easily recover optimal savings $(s_t^*)_{t\in\mathbb{T}}$ from (44).

2.2 Solving the problem by Lagrangian methods

- Under conditions satisfied here, Lagrangian techniques also applicable in infinite-dimensional cases (cf. Dechert (1982) or Le Van & Saglamb (2004))

- Define the Lagrangian function

$$\mathscr{L}((c_t, \mu_t)_{t\in\mathbb{T}}, \lambda) := \sum_{t\in\mathbb{T}} \Big[\beta^t u(c_t) + \mu_t c_t\Big] + \lambda M - \lambda \sum_{t\in\mathbb{T}} q_t c_t$$

- As in the finite-dimensional case, $(c_t^*)_{t\in\mathbb{T}} \geq 0$ solves (49) if there exist non-negative Lagrangian multipliers $(\mu_t^*)_{t\in\mathbb{T}}$ and $\lambda^* \geq 0$ such that the first order conditions (FOCs):

$$\frac{\partial \mathscr{L}}{\partial c_t}((c_t, \mu_t)_{t\in\mathbb{T}}, \lambda) = \beta^t u'(c_t) + \mu_t - \lambda q_t = 0 \quad \forall t \in \mathbb{T} \tag{50}$$

  the complementary slackness conditions (CSCs):

$$\mu_t c_t = \lambda M - \lambda \sum_{t\in\mathbb{T}} q_t c_t = 0 \tag{51}$$

  and (47) are satisfied for all t.

- Arguments analogous to Section 1.2 yield $\mu_t^* = 0$ for all $t \in \mathbb{T}$ and $\lambda^* > 0$.

Proposition 2.1 *Let Assumption 1.2 and the solvency condition (46) hold. Then, any sequence* $(c_t^*)_{t \in \mathbb{T}}$ *which solves*

$$\beta \frac{u'(c_{t+1})}{u'(c_t)} = \frac{q_{t+1}}{q_t} = \frac{1}{R_{t+1}} \quad \forall t \in \mathbb{T} \tag{52}$$

$$\sum_{t \in \mathbb{T}} q_t c_t = \sum_{t \in \mathbb{T}} q_t e_t + R_0 \bar{s}_{-1} \tag{53}$$

is a solution to (49).

- Equation (53) implies that optimal investment sequence $(s_t^*)_{t \in \mathbb{T}}$ defined by (44) satisfies

$$\lim_{t \to \infty} q_t s_t^* = 0 \tag{54}$$

- We will call (54) the transversality condition (TVC)
- As the optimal solution satisfies (52) for all t, $q_t = \beta^t u'(c_t^*)/u'(c_0^*)$
- As $u'(c_0^*)$ is just a constant, the TVC can equivalently be written as

$$\lim_{t \to \infty} \beta^t u'(c_t^*) s_t^* = 0 \tag{55}$$

- As (53) and (54) are equivalent, solution to original problem (43) can be characterized as:

Proposition 2.2 *Let Assumption 1.2 and the solvency condition (46) hold. Then, any sequence* $(c_t^*, s_t^*)_{t\in\mathbb{T}}$ *which satisfies*

$$\beta\frac{u'(c_{t+1})}{u'(c_t)} = \frac{q_{t+1}}{q_t} = \frac{1}{R_{t+1}} \tag{56}$$

$$c_t + s_t = e_t + R_t s_{t-1} \tag{57}$$

for all $t \in \mathbb{T}$ *(where* $s_{-1} = \bar{s}_{-1}$*) as well as the TVC (54) is a solution to (43).*

- Remark:
    - Although not obvious, the TVC is in fact a restriction on initial consumption c_0!
    - Will get back to this in Section 2.6

2.3 Solving the problem by variational methods

- Can also employ the following variational argument to obtain (52)
- Consider a variation of the optimal decision $(c_t^*)_{t\in\mathbb{T}}$ in period t_0 setting

$$\begin{aligned}\tilde{c}_{t_0} &= c_{t_0}^* - \delta \\ \tilde{c}_{t_0+1} &= c_{t_0+1}^* + \delta R_{t_0+1}\end{aligned}$$

  where δ is a small number.

- All other choices remain unchanged, can write utility as function H of δ:

$$H(\delta) := \beta^{t_0} u(c_{t_0}^* - \delta) + \beta^{t_0+1} u(c_{t_0+1}^* + \delta R_{t_0+1}) + \sum_{t\in\mathbb{T}\setminus\{t_0,t_0+1\}} \beta^t u(c_t^*). \tag{58}$$

- Since $(c_t^*)_{t\in\mathbb{T}}$ is optimal, H must be maximal for $\delta = 0$. This implies $H'(0) = 0$ which gives (52) for all t.

2.4 An equilibrium framework: The neoclassical growth model

- Embed previous problem into a dynamic macro-model with
  - a consumption sector consisting of N identical infinite-lived consumers
  - a production sector represented by a single firm

2.4.1 Consumption sector

- Each consumer:
  - planes over infinitely many future periods $\mathbb{T} = \{0, 1, 2, \ldots\}$ as in Section 2.1
  - supplies one unit of labor to the labor market in each period t
  - consumes c_t, invests s_t which becomes capital k_{t+1} in $t + 1$
  - capital earns return R_t, labor the wage w_t in t
- As consumers are identical, so are the decisions they take!

2.4.2 Production sector

- Production sector identical to Section 1.6.3, impose Assumption 1.3 on f
- Given labor $L_t = N$ and capital $K_t = Nk_t$, factor prices w_t and R_t determined by (33)

2.4.3 Consumer behavior

- Given her initial capital $k_0 > 0$, consumer chooses non-negative consumption-capital sequence (c_t, k_{t+1}) subject to budget constraint

$$k_{t+1} + c_t = w_t + k_t R_t, \quad \forall t \in \mathbb{T} \tag{59}$$

  to maximize utility $U((c_t)_{t\in\mathbb{T}})$ given by (42)

- Decision problem special case of (43) (where $e_t = w_t$, $s_t = k_{t+1}$, $\bar{s}_{-1} = k_0 > 0$)

- By Proposition 2.2, optimal decision characterized by (59), the Euler equations

$$\beta \frac{u'(c_{t+1})}{u'(c_t)} = R_{t+1}^{-1} \quad \forall t \in \mathbb{T} \tag{60}$$

  and the transversality condition

$$\lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = 0. \tag{61}$$

2.4.4 Equilibrium

- Economy is summarized by the list $\mathscr{E}_{NC} = \langle u, \beta, f \rangle$

Definition 2.1 *Given $k_0 > 0$, an equilibrium of $\mathscr{E}_{NC}$ is an allocation $(c_t^e, k_{t+1}^e)_{t\geq 0}$ and a price sequence $(w_t^e, R_t^e)_{t\geq 0}$ which satisfy (33), (59), (60), and (61)*

2.5 Equilibrium dynamics in state space form

- Equilibrium allocation solves the following implicit equations for all $t \geq 0$ by:

$$k_{t+1} + c_t - f(k_t) = 0 \tag{62a}$$
$$\beta u'(c_{t+1}) f'(k_{t+1}) - u'(c_t) = 0. \tag{62b}$$

- As u' is strictly decreasing and therefore, invertible, obtain explicit form of (62):

$$k_{t+1} = \varphi_k(k_t, c_t) := f(k_t) - c_t \tag{63a}$$
$$c_{t+1} = \varphi_c(k_t, c_t) := u'^{-1}\left(\frac{u'(c_t)}{\beta f'(f(k_t) - c_t)}\right). \tag{63b}$$

- System (63) represents the equilibrium dynamics in **state space form**
- $\varphi = (\varphi_k, \varphi_c)$ defined on

$$\mathbb{X} = \left\{(k, c) \in \mathbb{R}^2_{++} | c < f(k)\right\} \tag{64}$$

  but will see that $(\varphi, \mathbb{X})$ is not a dynamical system!

- Interior steady states $(\bar{k}, \bar{c}) \gg 0$ of (63) solve $f'(\bar{k}) = 1/\beta$ and $\bar{c} = f(\bar{k}) - \bar{k}$
- **Show:** Under Assumptions 1.2 and 1.3, there exists a unique interior steady state $(\bar{k}, \bar{c})$

2.6 A geometric interpretation of the transversality condition

2.6.1 An unstable steady state which is saddle-path stable

- System (63) offers nice geometric characterization of the TVC
- Will show this for special case where $u(c) = \log c$[1] and $f(k) = k^\alpha$, $0 < \alpha < 1$
- All qualitative insights extend to general case!
- Under previous parametrization, mapping φ in (63) reads:

$$\varphi_k(k, c) = k^\alpha - c \tag{65a}$$

$$\varphi_c(k, c) = \alpha\beta c \left(k^\alpha - c\right)^{\alpha - 1}. \tag{65b}$$

- Show (see Galor (2007) for additional details!):
  - system (65) has unique steady state $\bar{x} := (\bar{k}, \bar{c})$
  - Jacobian $D\varphi(\bar{x})$ has (real) Eigenvalues $|\lambda_1| < 1 < |\lambda_2|$
  - thus, $\bar{x}$ is unstable, in fact, *saddle-path stable* (cf. Problem 2.2 (iii) on PS 2!)
  - but: convergence towards $\bar{x}$ on lower-dimensional subset $\mathbb{M} \subset \mathbb{X}$ (stable manifold)

[1] Recall Section 1.5 and the remarks given there!

2.6.2 The stable manifold

- To determine $\mathbb{M}$, define $z_t := c_t/k_t^\alpha$ which evolves as

$$z_{t+1} = \psi(z_t) := \alpha\beta\frac{z_t}{1 - z_t} \tag{66}$$

  where ψ is defined on $]0,1[$, but $(\psi,]0,1[)$ is not a dynamical system!

- Dynamic properties of ψ [**show!**]:
  - ψ has unique non-trivial steady state $\bar{z} = 1 - \alpha\beta > 0$ which is unstable
  - for $z_0 < \bar{z}$, $\lim_{t\to\infty} \psi^t(z_0) = 0$ which implies $\lim_{t\to\infty} c_t = 0$ whenever $c_0 < \bar{z}k_0^\alpha$
  - for $z_0 > \bar{z}$, $\psi^{t_0}(z_0) > 1$ for finite $t_0 \geq 1$ which implies $k_{t_0+1} < 0$ whenever $c_0 > \bar{z}k_0^\alpha$

- Sequence $(k_t, c_t)_{t\geq 0}$ generated by (65) well-defined and does not diverge iff $c_0 = \bar{z}k_0^\alpha$

- Conclude that:
  - stable manifold is $\mathbb{M} = \{(k,c) \in \mathbb{R}^2_{++} | c = (1-\alpha\beta)k^\alpha\}$
  - equilibrium allocation satisfies $(k_t^e, c_t^e) \in \mathbb{M}$ for all $t \geq 0$ and $\lim_{t\to\infty}(k_t^e, c_t^e) = (\bar{k}, \bar{c})$
  - condition $c_0 = \bar{z}k_0^\alpha$ is equivalent to TVC (61) (cf. picture in class)
  - $\mathscr{E}_{NC}$ has a unique equilibrium which also holds in the general case!

2.7 A planning problem

- There is an alternative foundation for the equilibrium dynamics in the neoclassical model
- Consider a benevolent social planner who maximizes consumer utility by choosing a feasible allocation.

Definition 2.2 *Given $k_0 > 0$, a feasible allocation is a sequence $(c_t, k_{t+1})_{t \geq 0}$ which satisfies $c_t \geq 0$, $k_{t+1} \geq 0$ for all $t \geq 0$ as well as the resource constraint*

$$k_{t+1} + c_t \leq f(k_t). \tag{67}$$

The set of feasible allocations is denoted $\mathbb{A}(k_0)$.

- The planning problem reads:

$$\max_{(c_t, k_{t+1})_{t \in \mathbb{T}}} \left\{ U((c_t)_{t \in \mathbb{T}}) \,\middle|\, (c_t, k_{t+1})_{t \in \mathbb{T}} \in \mathbb{A}(k_0) \right\} \tag{68}$$

- Will see that the solution to (68) coincides with the equilibrium allocation!

2.8 Solving the planning problem by Lagrangian methods

- To solve (68), can use a variational argument as in Section 2.3 (or a Lagrangian approach as in Section 2.2) to obtain the conditions (62) for all $t \geq 0$.
- Will prove that if the solution to (62) satisfies the TVC (61), it solves the SPP (68).

Proposition 2.3 *Under Assumptions 1.2 and 1.3, the following holds for all* $k_0 > 0$*:*

(i) Any sequence $(c_t^*, k_{t+1}^*)_{t \geq 0}$ *which solves (62) for all* $t \geq 0$ *and (61) is a solution to (68).*

(ii) Any solution to (68) is unique.

Proof: Done in class.

- Previous result implies that equilibrium allocation is unique which in turn implies a unique equilibrium (**why?**)
- Solutions to (68) are precisely the Pareto-optimal allocations of $\mathscr{E}_{NC}$ (**why?**)
- Thus,there is an equivalence between *equilibrium* and *Pareto optimal allocations*
- Economically, this is a consequence of the first and second **Welfare Theorems**!
- Note that Proposition 2.3 does not, in general, deliver an existence result! This will be obtained next using recursive methods similar to Section 1.3.

2.9 Solving the planning problem by recursive methods

2.9.1 The Bellman equation

- Assume that f satisfies Assumption 1.3 and u the (weaker) Assumption 1.1 and $0 < \beta < 1$
- For each $k > 0$, define

$$V(k) := \sup_{(c_t, k_{t+1})_{t\in\mathbb{T}}} \Big\{ U((c_t)_{t\in\mathbb{T}}) \Big| (c_t, k_{t+1})_{t\in\mathbb{T}} \in \mathbb{A}(k) \Big\} \tag{69}$$

- We assume that the economy $\mathscr{E}$ is well-behaved such that $V(k) < \infty$ for all $k > 0$
- Principle of optimality implies that V solves functional equation (**Bellman-equation**):

$$V(k) = \max_{c,k_+} \Big\{ u(c) + \beta V(k_+) \Big| c \geq 0, k_+ \geq 0, k_+ + c \leq f(k) \Big\} \tag{70}$$

or, equivalently,

$$V(k) = \max_{k_+} \Big\{ u(f(k) - k_+) + \beta V(k_+) \Big| 0 \leq k_+ \leq f(k) \Big\} \tag{71}$$

- The solution V to (71) is called the **value function**

2.9.2 Existence of a solution to the Bellman equation

- For the following results, see Stokey & Lucas (1989) or Stachurski (2009)
- Following holds if u is bounded:
  - Bellman equation (70) has a unique solution $V : \mathbb{R}_+ \longrightarrow \mathbb{R}$ which is also bounded, strictly increasing, and strictly concave
  - V is a fixed point of an operator T which maps the space $\mathscr{C}(\mathbb{R}_+)$ of bounded continuous functions $G : \mathbb{R}_+ \longrightarrow \mathbb{R}$ into itself, i.e., $T : \mathscr{C}(\mathbb{R}_+) \longrightarrow \mathscr{C}(\mathbb{R}_+)$ and $TV = V$
  - T is a contraction on $\mathscr{C}(\mathbb{R}_+)$ which is a Banach space under the sup-norm.
  - By the *Contraction Mapping Theorem*, V is unique and $\lim_{n\to\infty} T^n G = V$ for all $G \in \mathscr{C}(\mathbb{R}_+)$ (where convergence is in the sup-norm)
  - This implies that V is continuous, increasing, and concave.
- Essentially same results hold if u not bounded but homogeneous of degree $\theta \leq 1$, cf. Alvarez & Stokey (1998) (as in Problems 1.1 and 3.1 which essentially have $\theta = 1 - \sigma$)

2.9.3 Policy function and an existence theorem

- Knowing V, can compute the **policy function** $\mathcal{K} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$,

$$\mathcal{K}(k) = \arg\max_{k_+} \left\{ u(f(k) - k_+) + \beta V(k_+) \,\middle|\, 0 \leq k_+ \leq f(k) \right\}. \tag{72}$$

  which determines optimal capital formation k_{t+1} in t as a function of current capital k_t.

- Policy function $\mathcal{K}$ is continuous by the *Theorem of the Maximum* (cf. Stokey & Lucas (1989, Theorem 3.6, p.62)) and satisfies $0 < \mathcal{K} < f$.

- Given $\mathcal{K}$, define the consumption function $\mathcal{C} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$,

$$\mathcal{C}(k) := f(k) - \mathcal{K}(k) \tag{73}$$

  which determines optimal consumption c_t in period t as a function of current capital k_t.

- Following is the main result of this section.

Proposition 2.4 *Let V be the solution to (70) and define $\mathcal{K}$ and $\mathcal{C}$ as in (72) and (73). Then, the sequence $(k^*_{t+1}, c^*_t)_{t\geq 0}$ defined recursively for $t \geq 0$ as:*

$$k^*_{t+1} = \mathcal{K}(k^*_t) \tag{74a}$$

$$c^*_t = \mathcal{C}(k^*_t) \tag{74b}$$

where $k^*_0 = k_0$ is a solution to (68).

2.9.4 Differentiability of the value function V

- Let u satisfy the stronger Assumption 1.2 and f satisfy Assumption 1.3 and $f(0) = 0$. These restrictions imply an interior solution to (71) for all $k > 0$.

- To recover first order conditions (62) of SPP (68) from (71) V would need to be differentiable to apply the envelope theorem as in Section 1.4.2

- As argued above, V obtains as the limit of continuous functions (under the sup norm) and is, therefore, continuous but need not be differentiable!

- However, can apply the *Beneviste-Scheinkman Theorem* to prove that V is differentiable (cf. Stokey & Lucas (1989), Theorems 4.10 and 4.11, pp.84/85) and satisfies for all $k > 0$:

$$V'(k) = u'(f(k) - \mathcal{K}(k))f'(k) \tag{75}$$

- In this case, $k_+ = \mathcal{K}(k)$ is determined by the first order conditions

$$u'(f(k) - k_+) + \beta V'(k_+) = 0. \tag{76}$$

- Combining (75) and (76) and setting $\mathcal{C}(k) := f(k) - \mathcal{K}(k)$ gives for all $k > 0$:

$$u'(\mathcal{C}(k)) + \beta f'(k_+)u'(\mathcal{C}(k_+)) = 0 \tag{77}$$

  which implies Euler equation (62b) when (77) is evaluated at optimal sequence $(k_t^*)_{t\geq 0}$ generated by (74a).

2.10 A beautiful result that connects the two approaches

- Following is the nexus between the

  1. *Lagrangian approach* from Section 2.2 which gave us the dynamics (63) in state space form
  2. *recursive approach* from Section 2.9 which gave us the policy functions (72) and (73)

- Let $\bar{x} = (\bar{k}, \bar{c})$ be the unique interior steady state of $\varphi = (\varphi_k, \varphi_c)$ from (63) defined on $\mathbb{X}$ as in (64). Define the stable manifold

$$\mathbb{M} := \Big\{ x = (k, c) \in \mathbb{X} \,|\, \varphi^t(k, c) \in \mathbb{X}\ \forall t \geq 0 \wedge \lim_{t\to\infty} \varphi^t(k, c) = (\bar{k}, \bar{c}) \Big\}. \tag{78}$$

- Let $\mathcal{K}$ be the policy defined in (72) and $\mathcal{C}$ the consumption function (73). Define its graph

$$\text{graph}(\mathcal{C}) := \Big\{ (k, c) \in \mathbb{X} \,|\, c = \mathcal{C}(k) \Big\}. \tag{79}$$

- Then, we have:

$$\mathbb{M} = \text{graph}(\mathcal{C}). \tag{80}$$

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