Mathematical Methods, Part 1:
Applied Intertemporal Optimization

Winter term 2015/16

Lecturer: Marten Hillebrand
(`mhillebr@uni-mainz.de`)

Teaching assistant: Van Hoang Khieu
(`hkhieu@uni-mainz.de`)

Organizational issues

- Part 1:
    - seven classes:
        - 10:15 - 11:45 h
        - 12:30 - 14:00 h
    - midterm exam on Nov. 30
- Problem sets:
    - issued every week on Mondays
    - due next Monday, to be handed-in prior to class!
- Main reference:
    - slides (shortly available via OLAT)
    - book '*Applied Intertemporal Optimization*' by Klaus Wälde (freely available at `http:///www.waelde.com`)
    - additional references on the slides
- Answer all (explicit or implicit) questions on slides, prove all results!

Objective

Part 1 of his course is on *Applied Intertemporal Optimization*. Its general aim is to provide participants with the tools and mathematical methods necessary to analyze and solve optimization problems and dynamic models in discrete and continuous time and in deterministic and stochastic environments. Problems and models of this type constitute a major building block of modern macroeconomic theory and many other areas such as finance, etc. The theory presented in class is complemented by problem sets which serve to illustrate and amplify the theoretical results and their applications.

Contents

I Deterministic Models in Discrete Time **10**

1 Decision problems with finite time horizon **11**

1.1 A prototype consumption investment problem 11

1.1.1 Setup 11

1.1.2 Intertemporal budget set 12

1.1.3 Preferences and decision problem 14

1.2 Solving the problem by Lagrangian methods 15

1.3 Solving the problem by recursive methods 17
1.3.1 The three-period case 17
1.3.2 The general multi-period case 18
1.4 Characterizing the recursive solution by first order conditions 19
1.4.1 Differentiability of the value functions 19
1.4.2 The envelope theorem 20
1.5 Handling problems with unbounded utility 21
1.6 An equilibrium framework: The OLG model 22
1.6.1 Population structure 22
1.6.2 Consumer behavior 23
1.6.3 Production side 25
1.6.4 Equilibrium 26
1.6.5 Recursive structure of equilibrium 27
1.7 Dynamical systems in discrete time 29
1.7.1 State space, time-one map, orbits 29
1.7.2 Fixed points and stability 30
1.7.3 Analyzing the dynamics with phase diagrams 32
1.7.4 The Grobman-Hartman theorem 33
1.8 Equilibrium dynamics in the OLG model 35
1.8.1 Existence and non-existence of steady states 35
1.8.2 Uniqueness and multiplicity of steady states 36

2 Decision problems with infinite time horizon **37**
2.1 A prototype consumption investment problem 37
2.1.1 Decision setup 37
2.1.2 NPG-condition and intertemporal budget set 38
2.1.3 Preferences and decision problem 39
2.1.4 Lifetime budget constraint 40
2.2 Solving the problem by Lagrangian methods 42
2.3 Solving the problem by variational methods 45
2.4 An equilibrium framework: The neoclassical growth model 46
2.4.1 Consumption sector 46
2.4.2 Production sector 46
2.4.3 Consumer behavior 47
2.4.4 Equilibrium 47
2.5 Equilibrium dynamics in state space form 48
2.6 A geometric interpretation of the transversality condition 49
2.6.1 An unstable steady state which is saddle-path stable 49
2.6.2 The stable manifold 50
2.7 A planning problem 51
2.8 Solving the planning problem by Lagrangian methods 52
2.9 Solving the planning problem by recursive methods 53
2.9.1 The Bellman equation 53

2.9.2 Existence of a solution to the Bellman equation 54
2.9.3 Policy function and an existence theorem 55
2.9.4 Differentiability of the value function V 56
2.10 A beautiful result that connects the two approaches 57

II Stochastic Models in Discrete Time 59

3 Setting up a random environment 60
3.1 Basic concepts from probability theory 61
3.1.1 Probability spaces and random variables 61
3.1.2 Probability and distributions of random variables 62
3.2 Constructing the underlying probability space 63
3.2.1 Example 1: Independent random variables 63
3.2.2 Example 2: Correlated random variables 64
3.3 Filtration and conditional expectation 65

4 Stochastic decision problems with finite horizon 66
4.1 A stochastic OLG model 66
4.1.1 Production side 67
4.1.2 A stochastic two-period decision problem 68

4.1.3 Deriving the equilibrium equations 70
4.1.4 Equilibrium 71
4.1.5 Recursive structure of equilibrium 72

5 Stochastic dynamical systems in discrete time **74**
5.1 Stochastic dynamical systems and stability 74
5.2 Markov operator 75
5.3 Stochastic steady states 76
5.4 Equilibrium dynamics in the stochastic OLG model 77
5.4.1 Stable sets 77
5.4.2 Existence of a stochastic steady state 78

6 Stochastic decision problems with infinite horizon **79**
6.1 A prototype decision problem 79
6.1.1 Decision setup 80
6.1.2 Intertemporal budget set 81
6.1.3 Preferences and decision problem 82
6.2 Solving the decision problem 83
6.3 An equilibrium framework: The RBC model 85
6.3.1 Consumer side 86
6.3.2 Production side 88

6.4 Equilibrium 90
6.5 A stochastic planning problem 91
6.6 Solving the stochastic planning problem by recursive methods 92
6.6.1 The Bellman equation 92
6.6.2 Policy function 93
6.7 Equilibrium dynamics in the RBC model 94

References

Alvarez, F. & N. L. Stokey (1998): "Dynamic Programming with Homogeneous Functions", *Journal of Economic Theory*, 82, 167–189.

Arnold, L. (1998): *Random Dynamical Systems*. Springer-Verlag, Berlin a.o.

Brock, W. A. & L. J. Mirman (1972): "Optimal Growth and Uncertainty: The Discounted Case", *Journal of Economic Theory*, 4, 479–513.

Dechert, W. (1982): "Lagrange multipliers in infinite horizon discrete time optimal control models", *Journal of Mathematical Economics*, 9, 285–302.

Galor, O. (2007): *Discrete Dynamical Systems*. Springer, Berlin, a.o.

Galor, O. & H. E. Ryder (1989): "Existence, Uniqueness, and Stability of Equilibrium in an Overlapping–Generations Model with Productive Capital", *Journal of Economic Theory*, 49, 360–375.

Hillebrand, M. (2008): *Pension Systems, Demographic Change, and the Stock Market*, Bd. 610 of *Lecture Notes in Economics and Mathematical Systems*. Springer, Berlin, Heidelberg, New York.

Kamihigashi, T. & J. Stachurski (2014): "Stochastic Stability in Monotone Economies", *Theoretical Economics*, 9, 383 – 407.

Le Van, C. & H. C. Saglamb (2004): "Optimal growth models and the Lagrange multiplier", *Journal of Mathematical Economics*, 40, 393410.

Mas-Colell, A., M. D. Whinston & J. R. Green (1995): *Microeconomic Theory*. Oxford University Press, Oxford a.o.

McGovern, J., O. F. Morand & K. L. Reffett (2013): "Computing minimal state space recursive equilibrium in OLG models with stochastic production", *Economic Theory*, 54, 623–674.

Morand, O. F. & K. L. Reffett (2007): "Stationary Markovian Equilibrium in Overlapping Generations Models with Stochastic Nonclassical Production and Markov Shocks", *Journal of Mathematical Economics*, 43, 501–522.

Schenk-Hoppé, K. R. & B. Schmalfuss (2001): "Random fixed points in a stochastic Solow growth model.", *Journal of Mathematical Economics*, 36, 19–30.

Stachurski, J. (2009): *Economic Dynamics: Theory and Computation*. MIT Press, Cambridge (Mass.) a.o.

Stokey, N. L. & R. E. J. Lucas (1989): *Recursive Methods in Economic Dynamics*. Harvard University Press, Cambridge (Mass.) a.o.

Wang, Y. (1993): "Stationary Equilibria in an Overlapping Generations Economy with Stochastic Production", *Journal of Economic Theory*, 61(2), 423–435.