Mathematical Methods, Part 1: Applied Intertemporal Optimization

Winter term 2015/16

Lecturer: Marten Hillebrand
(mhillebr@uni-mainz.de)
Teaching assistant: Van Hoang Khieu
(hkhieu@uni-mainz.de)

Organizational issues

- Part 1:
  - seven classes:
    - 10:15 - 11:45 h
    - 12:30 - 14:00 h
  - midterm exam on Nov. 30
- Problem sets:
  - issued every week on Mondays
  - due next Monday, to be handed-in prior to class!
- Main reference:
  - slides (shortly available via OLAT)
  - book '*Applied Intertemporal Optimization*' by Klaus Wälde (freely available at `http:///www.waelde.com`)
  - additional references on the slides
- Answer all (explicit or implicit) questions on slides, prove all results!

Objective

Part 1 of his course is on *Applied Intertemporal Optimization*. Its general aim is to provide participants with the tools and mathematical methods necessary to analyze and solve optimization problems and dynamic models in discrete and continuous time and in deterministic and stochastic environments. Problems and models of this type constitute a major building block of modern macroeconomic theory and many other areas such as finance, etc. The theory presented in class is complemented by problem sets which serve to illustrate and amplify the theoretical results and their applications.

Contents

Mathematical Methods, Part 1: Applied Intertemporal Optimization

Part I

Winter term 2015/16

Lecturer: Marten Hillebrand
Teaching assistant: Van Hoang Khieu

Part I
Deterministic Models in Discrete Time

1 Decision problems with finite time horizon

1.1 A prototype consumption investment problem

1.1.1 Setup

- Consider single consumer/household:
  - takes decisions in *initial period* $t = 0$
  - life/planning horizon ends in *terminal period* $t = T > 0$
  - *planing horizon* is $\mathbb{T} := \{0, 1, \ldots, T\}$
- Single consumption good ('numeraire'), all quantities denominated in consumption units
- In each period $t \in \mathbb{T}$:
  - consumer earns exogenous non-capital income $e_t \geq 0$
  - consumes $c_t \geq 0$, invests/borrows capital s_t
  - one unit invested in $t - 1$ earns gross return $R_t > 0$ in t
- Thus, to make problem more interesting, allow for unbounded loans
- But: require $s_T \geq 0$, i.e., no loans in terminal period

1.1.2 Intertemporal budget set

- Given quantities in decision:
  - non-capital incomes $e^T := (e_0, \ldots, e_T)$
  - returns $R^T := (R_0, \ldots, R_T)$
  - initial capital $\bar{s}_{-1}$
- Decision variables:
  - consumption plan $(c_t)_{t \in \mathbb{T}}$
  - investment plan $(s_t)_{t \in \mathbb{T}}$
- Sequential budget constraint for all $t \in \mathbb{T}$:

$$c_t + s_t \leq e_t + R_t s_{t-1} \tag{1}$$

  where $s_{-1} \equiv \bar{s}_{-1}$

- Incomes, returns, and initial capital determine budget set:

$$\mathbb{B}(e^T, R^T, \bar{s}_{-1}) := \Big\{(c_t, s_t)_{t \in \mathbb{T}} | c_t \geq 0, (1) \text{ holds for all } t \in \mathbb{T}, s_T \geq 0\Big\} \tag{2}$$

- Investment sequence $(s_t)_{t\in\mathbb{T}}$ determines future wealth levels

$$W_t := e_t + R_t s_{t-1}, \quad t > 0. \tag{3}$$

- Define discounted future lifetime income

$$E_t := \frac{e_{t+1}}{R_{t+1}} + \ldots + \frac{e_T}{R_{t+1}\cdots R_T} = \sum_{n=t+1}^{T} e_n \prod_{m=t+1}^{n} R_m^{-1} \tag{4}$$

for $t \in \mathbb{T}\backslash\{T\}$ and $E_T := 0$.

Lemma 1.1 *The budget set defined in (2) satisfies the following:*

(i) Any $(c_t, s_t)_{t\in\mathbb{T}} \in \mathbb{B}(e^T, R^T, \bar{s}_{-1})$ *satisfies*

$$s_t \geq \underline{s}_t := -E_t \tag{5}$$

for all $t \in \mathbb{T}$ *while the wealth levels defined in (3) satisfy* $W_t \geq -E_t$*.*

(ii) $\mathbb{B}(e^T, R^T, \bar{s}_{-1})$ *is non-degenerate (contains more than one element) iff*

$$\bar{s}_{-1} > -\frac{e_0 + E_0}{R_0}. \tag{6}$$

(iii) $\mathbb{B}(e^T, R^T, \bar{s}_{-1})$ *is compact and convex* $\forall (e^T, R^T, \bar{s}_{-1}) \in \mathbb{R}_+^{T+1} \times \mathbb{R}_{++}^{T+1} \times \mathbb{R}$ *satisfying (6).*

- Remainder assumes that solvency condition (6) holds.

1.1.3 Preferences and decision problem

- Consumer has time-additive utility function $U : \mathbb{R}_+^{T+1} \longrightarrow \mathbb{R}$,

$$U((c_t)_{t\in\mathbb{T}}) = \sum_{t=0}^{T} \beta^t u(c_t), \quad \beta > 0. \tag{7}$$

- Decision problem reads:

$$\max\Big\{U((c_t)_{t\in\mathbb{T}})|(c_t, s_t)_{t\in\mathbb{T}} \in \mathbb{B}(e^T, R^T, \bar{s}_{-1})\Big\}. \tag{8}$$

- **Show:** Following restriction on u sufficient for (8) to have a unique solution:

  Assumption 1.1 *The period utility function $u : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is continuous, strictly increasing, and strictly concave.*

- Following stronger restriction will be convenient more:

  Assumption 1.2 *The period utility function $u : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is continuous and C^2 on $\mathbb{R}_{++}$ with derivatives satisfying $u'' < 0 < u'$ and the Inada condition $\lim_{c\searrow 0} u'(c) = \infty$.*

1.2 Solving the problem by Lagrangian methods

- Adopt a standard Lagrangian approach to solve (8)
- Let u satisfy stronger Assumption 1.2 and let solvency condition (6) hold
- Define the Lagrangian function

$$\mathscr{L}((c_t, s_t, \mu_t, \lambda_t)_{t \in \mathbb{T}}) := \sum_{t \in \mathbb{T}} \left[\beta^t u(c_t) + \mu_t c_t + \lambda_t (e_t + R_t s_{t-1} - s_t - c_t) \right]$$

- Standard arguments imply that $(c_t^*, s_t^*)_{t \in \mathbb{T}}$ solves (8) if there exist non-negative Lagrangian multipliers $(\mu_t^*, \lambda_t^*)_{t \in \mathbb{T}}$ such that $(c_t^*, s_t^*, \mu_t^*, \lambda_t^*)_{t \in \mathbb{T}}$ solves the first order conditions (FOCs):

$$\frac{\partial \mathscr{L}}{\partial c_t}((c_t, s_t, \mu_t, \lambda_t)_{t \in \mathbb{T}}) = \beta^t u'(c_t) + \mu_t - \lambda_t = 0 \quad \forall t \in \mathbb{T} \tag{9}$$

$$\frac{\partial \mathscr{L}}{\partial s_t}((c_t, s_t, \mu_t, \lambda_t)_{t \in \mathbb{T}}) = -\lambda_t + R_{t+1} \lambda_{t+1} = 0 \quad \forall t \in \mathbb{T} \backslash \{T\} \tag{10}$$

  the complementary slackness conditions (CSCs):

$$\mu_t c_t = \lambda_t (e_t + R_t s_{t-1} - s_t - c_t) = 0 \tag{11}$$

  and (1) and $c_t \geq 0$ for all $t \in \mathbb{T}$ where $s_{-1} = \bar{s}_{-1}$ and $s_T = 0$.

- Observations:
  - boundary behavior of u' from Assumption 1.2 excludes $c_t^* = 0$
  - thus, $\mu_t^* = 0$ for all $t \in \mathbb{T}$ by (11)
  - then, by (9) and (10), $\lambda_t^* > 0$
  - hence, (1) is binding for all $t \in \mathbb{T}$ by (11).
- These results give:

  Proposition 1.1 *Let Assumption 1.2 and the solvency condition (6) be satisfied. Then, any list* $(c_t^*, s_t^*)_{t\in\mathbb{T}}$ *which solves the conditions*

$$\beta u'(c_{t+1})R_{t+1} = u'(c_t) \qquad \forall t \in \mathbb{T}\backslash\{T\} \tag{12a}$$

$$c_t + s_t = e_t + R_t s_{t-1} \quad \forall t \in \mathbb{T} \tag{12b}$$

$$s_T = 0 \tag{12c}$$

  is a solution to (8).
- Interpretation of (12a) ('MRS' = 'marginal rate of substitution'):

$$\underbrace{\frac{\beta u'(c_{t+1})}{u'(c_t)}}_{\text{intertemporal MRS}} = \underbrace{\frac{1}{R_{t+1}}}_{\text{intertemporal price ratio}}$$

1.3 Solving the problem by recursive methods

1.3.1 The three-period case

- Impose [weaker] Assumption 1.1 on u, suppose first $T = 2$
- Consider future decision problem in $t = T-1 = 1$ given arbitrary $W_1 = e_1 + R_1 s_0 \geq -E_1$:

$$\max_{c_1, s_1} \Big\{ u(c_1) + \beta u(e_2 + s_1 R_2) | c_1 \geq 0, c_1 + s_1 \leq W_1, s_1 \geq -E_1 \Big\}$$

- For each $W \geq -E_1$, define the *value function* [**why well-defined?**]

$$V_1(W) := \max_{c_1, s_1} \Big\{ u(c_1) + \beta u(e_2 + s_1 R_2) | c_1 \geq 0, c_1 + s_1 \leq W, s_1 \geq -E_1 \Big\}.$$

- **Principle of Optimality** states that

$$\begin{aligned} & \max_{(c_t, s_t)_{t \in \mathbb{T}}} \Big\{ U((c_t)_{t \in \mathbb{T}}) | (c_t, s_t)_{t \in \mathbb{T}} \in \mathbb{B}(e^T, R^T, \bar{s}_{-1}) \Big\} \\ = \quad & \max_{c_0, s_0} \Big\{ u(c_0) + \beta V_1(e_1 + s_0 R_1) \,|\, c_0 \geq 0, s_0 \geq -E_0, c_0 + s_0 \leq e_0 + R_0 \bar{s}_{-1} \Big\}. \quad (13) \end{aligned}$$

- Knowing V_1, obtain optimal decision (c_0^*, s_0^*) for $t = 0$ by solving one-stage problem (13)!

1.3.2 The general multi-period case

- Straightforward to generalize previous approach.

- Define value functions $(V_t)_{t\in\mathbb{T}}$ recursively by setting $V_T = u$ and, for all $W \geq -E_t$:

$$V_t(W) = \max_{c,s}\Big\{u(c) + \beta V_{t+1}(e_{t+1} + sR_{t+1}) \,|\, c \geq 0, s \geq -E_t, s + c \leq W\Big\} \tag{14}$$

- **Prove**: Under Assumption 1.1, each function V_t, $t \in \mathbb{T}$ is well-defined and continuous, strictly increasing, and strictly concave.

- Obtain the optimal decision in $t = 0$ as:

$$(c_0^*, s_0^*) = \arg\max_{c,s}\Big\{u(c) + \beta V_1(e_1 + sR_1) \,|\, c \geq 0, s \geq -E_0, c + s \leq e_0 + R_0\bar{s}_{-1}\Big\}. \tag{15}$$

- Can recover optimal decision for all $t \in \mathbb{T}$ recursively by setting $W_t^* = e_t + s_{t-1}^* R_t$ and

$$(c_t^*, s_t^*) = \arg\max_{c,s}\Big\{u(c) + \beta V_{t+1}(e_{t+1} + sR_{t+1}) \,|\, c \geq 0, s \geq -E_t, c + s \leq W_t^*\Big\}. \tag{16}$$

- Remark: A rigorous proof of the principle of optimality in a related context can be found in Hillebrand (2008).

1.4 Characterizing the recursive solution by first order conditions

1.4.1 Differentiability of the value functions

- Restrict u by stronger Assumption 1.2
- In this case, any candidate solution to (8) satisfies $W_t > -E_t$ for all $t \in \mathbb{T}$
- Using simple induction and the *implicit function theorem* (cf. Mas-Colell, Whinston & Green (1995), Appendix M.E), one (**you!** :-)) can show that
  - each value function V_t, $t \in \mathbb{T}$ is C^1
  - the solution (c_t^*, s_t^*) to (14) is determined by C^1 functions

$$\begin{array}{rclr} C_t :]-E_t, \infty[& \longrightarrow & \mathbb{R}_{++}, & c_t^* = C_t(W_t) \\ S_t :]-E_t, \infty[& \longrightarrow &]-E_t, \infty[, & s_t^* = S_t(W_t). \end{array}$$

- Using this in (14) and the budget constraint gives for all $t < T$ and $W > -E_t$:

$$V_t(W) = u(W - S_t(W)) + \beta V_{t+1}(e_{t+1} + R_{t+1} S_t(W)) \tag{17}$$

- Also note that the optimal solution $S_t(W)$ satisfies the FOC's [**why?**]

$$-u'(W - S_t(W)) + \beta R_{t+1} V'_{t+1}(e_{t+1} + R_{t+1} S_t(W)). \tag{18}$$

1.4.2 The envelope theorem

- Differentiating (17) using (18) gives for all $t < T$ and $W > -E_t$:

$$V_t'(W) = u'(W - S_t(W)) = u'(C_t(W)). \tag{19}$$

- This is nothing but a simple application of the *envelope theorem* (see Mas-Colell, Whinston & Green (1995), Appendix M.L)

- We now clam that for all t and $W > -E_t$:

$$- u'(W - S_t(W)) + \beta R_{t+1} u'(e_{t+1} + R_{t+1} S_t(W)). \tag{20}$$

- To see this, suppose first $t = T - 1$. Then, $V_{t+1}' = u'$ can be used in (18) to obtain (20).

- Second, suppose $t < T - 1$. Then, (19) gives $V_{t+1}'(W) = u'(C_{t+1}(W))$ for all $W > -E_{t+1}$. Using this in (18) also gives (20).

- Defining the optimal wealth levels $(W_t^*)_{t \in \mathbb{T}}$ recursively by (3) and setting $c_t^* := C_t(W_t^*)$, equation (20) can be written as

$$- u'(c_t^*) + \beta R_{t+1} u'(c_{t+1}^*). \tag{21}$$

  which is precisely the optimality condition (12a).

1.5 Handling problems with unbounded utility

- Some popular utility functions including log-utility satisfy $\lim_{c \searrow 0} u(c) = -\infty$
- Problem:
  - u not defined for $c = 0$
  - U not continuous on budget set $\rightsquigarrow$ above's existence argument fails!
- Thus, these functions are excluded by Assumption 1.1
- But: Can handle problem as follows:
  - choose a (very small) lower bound $\underline{c} > 0$
  - add restriction $c_t \geq \underline{c}$ to budget set (2)
  - choosing $\underline{c} > 0$ small enough ensures that $c_t^* > \underline{c}$ for all $t \in \mathbb{T}$
- Remark: Previous modification changes lower bounds on savings to

$$\underline{s}_t \geq -\hat{E}_t := -\left[\frac{e_{t+1} - \underline{c}}{R_{t+1}} + \ldots + \frac{e_T - \underline{c}}{R_{t+1} \cdots R_T}\right] = -\sum_{n=t+1}^{T} (e_n - \underline{c}) \prod_{m=t+1}^{n} R_m^{-1} \tag{22}$$

- If (6) holds for $\underline{c} = 0$, will continue to hold for $\underline{c} > 0$ small!

1.6 An equilibrium framework: The OLG model

1.6.1 Population structure

- Growth model with overlapping generations (OLG) of consumers provides natural framework for intertemporal decision problems with finite horizon
- Consider simplest case with stationary population of two-period lived consumers:
  - new generation of $N \geq 0$ consumers born in each period $t \geq 0$
  - these consumers live for two periods, die at end of $t+1$
  - generational index $j \in \{y, o\}$ identifies 'young' and 'old' generation in $t \geq 0$
- Young consumer in period t:
  - supplies one unit of labor
  - saves/invests s_t
  - consumes c_t^y
- Old consumer in period t:
  - supplies capital $k_t = s_{t-1}$
  - consumes c_t^o

1.6.2 Consumer behavior

- A young consumer in period $t \geq 0$
  - labor income $w_t > 0$ in t, no labor income in $t+1$
  - chooses savings $s_t \geq 0$ and lifetime consumption $(c_t^y, c_{t+1}^o) \geq 0$ subject to:

$$c_t^y = w_t - s_t \tag{23a}$$

$$c_{t+1}^o = R_{t+1} s_t \tag{23b}$$

  - lifetime utility function $U(c^y, c^o) := u(c^y) + \beta u(c^o)$, $\beta > 0$
  - decision problem:

$$\max_s \left\{ u(w_t - s) + \beta u(sR_{t+1}) \mid 0 \leq s \leq w_t \right\} \tag{24}$$

- Impose Assumption 1.2 on utility u.

- Optimal decision:
  - special case of decision problem ($T = 1$, $e_0 = w_t$, $e_1 = 0$)
  - unique solution determined by first order conditions

Lemma 1.2 *Let Assumption 1.2 be satisfied. Then, for each $(w_t, R_{t+1}) \gg 0$, there exists a unique solution s_t to (24) determined by*

$$u'(w_t - s_t) - R_{t+1}\,\beta u'(s_t R_{t+1}) = 0. \tag{25}$$

- Aggregate investment made at time t determines next periods's capital stock:

$$K_{t+1} = N s_t \tag{26}$$

- Defining per-capita capital stock $k_t := K_t/N$, (25) can be written as:

$$u'(w_t - k_{t+1}) - R_{t+1}\,\beta u'(k_{t+1} R_{t+1}) = 0. \tag{27}$$

- Consumption in t satisfies:

$$c_t^y = w_t - k_{t+1}. \tag{28}$$

- Old consumer in period $t \geq 0$ consumes his entire (capital) income:

$$c_t^o = R_t k_t. \tag{29}$$

1.6.3 Production side

- Representative firm produces output Y using labor and capital as inputs:

$$Y = F(K, L) \tag{30}$$

- Linear homogeneous technology F can be written as:

$$Y = Lf(K/L) \quad \text{where} \quad f(k) := F(k, 1) \tag{31}$$

- Remark: Interpret f as a *gross* production function that includes non-depreciated capital

Assumption 1.3 *The intensive form production function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is C^2 with derivatives satisfying $f'' < 0 < f'$, $\lim_{k \to 0} f'(k) = \infty$, and $\lim_{k \to \infty} f'(k) < 1$.*

- Given $(w_t, R_t) \gg 0$, firm chooses input demand to maximize profits:

$$\max_{K,L} \left\{ Lf\left(\frac{K}{L}\right) - R_t K - w_t L \mid (K, L) \in \mathbb{R}_+^2 \right\} \tag{32}$$

- FOCs of (32) determine equilibrium factor prices as function of capital intensity $k_t = \frac{K_t}{L_t}$:

$$w_t = \mathcal{W}(k_t) := f(k_t) - k_t f'(k_t) \tag{33a}$$

$$R_t = \mathcal{R}(k_t) := f'(k_t) \tag{33b}$$

1.6.4 Equilibrium

- OLG economy is summarized by the list $\mathscr{E}_{\text{OLG}} = \langle u, \beta, f \rangle$
- Following definition of equilibrium is standard

Definition 1.1 *Given* $k_0 > 0$*, an equilibrium of* $\mathscr{E}_{\text{OLG}}$ *consists of prices* $(w_t^e, R_t^e)_{t \geq 0}$ *and an allocation* $(k_{t+1}^e, c_t^{y,e}, c_t^{o,e})_{t \geq 0}$ *which satisfy equations (27), (28), (29), and (33) for all* $t \geq 0$.

- Questions:
  - existence of equilibrium?
  - uniqueness of equilibrium?
  - dynamic behavior of equilibrium?
- To answer them, will exploit recursive structure of equilibrium derived next

1.6.5 Recursive structure of equilibrium

- Given $k > 0$, define for each $0 < k_+ < \mathcal{W}(k)$ the function

$$H(k_+; k) := u'(\mathcal{W}(k) - k_+) - \beta\mathcal{R}(k_+)\, u'(k_+\mathcal{R}(k_+)). \tag{34}$$

- Equilibrium sequence $(k^e_{t+1})_{t\geq 0}$ satisfies $H(k^e_{t+1}; k^e_t) = 0$ for all $t \geq 0$ and determines all other equilibrium variables

- Uniqueness result derived below requires either of the following additional restriction:

  Assumption 1.4 *(a) The production function f satisfies $\frac{kf''(k)}{f'(k)} \geq -1$ for all $k > 0$.*

  (b) The utility function u satisfies $\frac{cu''(c)}{u'(c)} \geq -1$ for all $c > 0$.

- **Prove** the following auxiliary result:

  Lemma 1.3 *Under Assumptions 1.2 and 1.3, the following holds:*

  (i) The function $H(\cdot; k)$ defined in (34) has at least one zero for all $k > 0$.

  (ii) If, in addition either (a) or (b) of Assumption 1.4 hold, this zero is unique.

- Lemma 1.3 allows us to state the following main result:

  Proposition 1.2 *Under Assumptions 1.2 and 1.3, the following holds for all* $k_0 > 0$*:*

  (i) Economy $\mathscr{E}_{\mathrm{OLG}}$ *has at least one equilibrium.*

  (ii) If, in addition, either (a) or (b) of Assumption 1.4 hold, this equilibrium is unique.

- Observations:
  - additional restrictions in Assumption 1.4 ensure existence of a map $\mathcal{K} : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ which determines the unique solution $k_+ = \mathcal{K}(k)$ to (34) for each $k > 0$
  - by the implicit function theorem, $\mathcal{K}$ is C^1, strictly increasing, and satisfies

$$0 < \mathcal{K}(k) < \mathcal{W}(k) < f(k) \tag{35}$$

  - unique equilibrium sequence $(k^e_{t+1})_{t\geq 0}$ determined recursively by $k^e_0 = k_0$ and

$$k^e_{t+1} = \mathcal{K}(k^e_t), \quad t \geq 0. \tag{36}$$

- To study equilibrium dynamics, need basic concepts from dynamical systems theory.

1.7 Dynamical systems in discrete time

1.7.1 State space, time-one map, orbits

- Let $\mathbb{X} \subset \mathbb{R}^N$ be
  - a non-empty (typically: open/closed, convex) set
  - endowed with Euclidean norm $\|\cdot\|$ and relative topology
- Let $\varphi : \mathbb{X} \longrightarrow \mathbb{X}$ be a function which maps $\mathbb{X}$ into itself
- $(\varphi, \mathbb{X})$ is a *deterministic dynamical system* with *time-one-map* φ and *state space* $\mathbb{X}$
- Each $x_0 \in \mathbb{X}$ induces a sequence $\{x_t\}_{t\geq 0}$ defined recursively as

$$x_{t+1} = \varphi(x_t) = \varphi(\varphi(x_{t-1})) = \varphi^2(x_{t-1}) = \varphi^{t+1}(x_0), \quad t > 0.$$

where $\varphi^t = \underbrace{\varphi \circ \ldots \circ \varphi}_{t\text{ - times}} : \mathbb{X} \longrightarrow \mathbb{X}$, $t > 0$ and $\varphi^0 := \mathrm{id}_{\mathbb{X}}$

- For each $x_0 \in \mathbb{X}$, call

$$\Gamma(x_0) := (\varphi^t(x_0))_{t\geq 0}$$

the *orbit* of x_0 (under φ)

1.7.2 Fixed points and stability

- Of particular interest: Points $\bar{x} \in \mathbb{X}$ which are limits of orbits and for which the dynamics become 'steady'

Definition 1.2 *A fixed point of dynamical system* $(\mathbb{X}, \varphi)$ *is a value* $\bar{x} \in \mathbb{X}$ *for which* $\bar{x} = \varphi(\bar{x})$.

- Orbit $\Gamma(\bar{x}) = (\bar{x}, \bar{x}, \ldots)$ is the constant sequence

- For which initial values $x_0 \in \mathbb{X}$ does $\Gamma(x_0)$ converge to fixed point $\bar{x}$, i.e.,

$$\lim_{t\to\infty} \varphi^t(x_0) = \bar{x}. \tag{37}$$

Definition 1.3 *A fixed point* $\bar{x} \in \mathbb{X}$ *of a dynamical system* $(\varphi, \mathbb{X})$ *is called*

(i) locally asymptotically stable, if there exists a neighborhood $U \subset \mathbb{X}$ *containing* $\bar{x}$ *such that* $\lim_{t\to\infty} \|\varphi^t(x) - \bar{x}\| = 0$ *for all* $x \in U$.

(ii) globally asymptotically stable if $\lim_{t\to\infty} \|\varphi^t(x) - \bar{x}\| = 0$ *for all* $x \in \mathbb{X}$.

(iii) unstable if it is not locally asymptotically stable.

- Clear:
  - global stability implies local stability
  - global stability requires φ to have a *unique* fixed point in $\mathbb{X}$
- Terminology:
  - unless stated otherwise, 'stable' taken to mean 'locally asymptotically stable'
  - fixed points synonymously referred to as 'steady states'

1.7.3 Analyzing the dynamics with phase diagrams

- done in class.

1.7.4 The Grobman-Hartman theorem

- Assume:
  - state space $\mathbb{X} \subset \mathbb{R}^N$ is an open set
  - time-one map φ is continuously differentiable (C^1)
- Next result:
  - sufficient criterion to infer local stability of $\bar{x}$ from Jacobian matrix $D\varphi(\bar{x})$
  - write $\varphi = (\varphi^{(1)}, \dots, \varphi^{(N)})$ where $\varphi^{(n)} : \mathbb{X} \longrightarrow \mathbb{R}$, $n = 1, \dots, N$

Lemma 1.4 (Grobman-Hartman Theorem) *Let φ be C^1 and $\bar{x} \in \mathbb{X}$ be a steady state of φ. Denote by $\lambda_n \in \mathbb{C}$, $n = 1, \dots, N$ the Eigenvalues of the Jacobian matrix*

$$D\varphi(\bar{x}) = \begin{bmatrix} \frac{\partial \varphi^{(1)}}{\partial x^{(1)}}(\bar{x}) & \dots & \frac{\partial \varphi^{(1)}}{\partial x^{(N)}}(\bar{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi^{(N)}}{\partial x^{(1)}}(\bar{x}) & \dots & \frac{\partial \varphi^{(N)}}{\partial x^{(N)}}(\bar{x}) \end{bmatrix} \in \mathbb{R}^{N \times N}.$$

Then the following holds true:

(i) If $|\lambda_n| < 1$ for all $n \in \{1, \dots, N\}$, then $\bar{x}$ is locally asymptotically stable.

(ii) If $|\lambda_n| > 1$ for at least one $n \in \{1, \dots, N\}$, then $\bar{x}$ is unstable.

- In this course, we will mostly consider models where $N = 1$ in which case $\mathbb{X} \subset \mathbb{R}$ and the dynamics are one-dimensional

- A straightforward corollary to Lemma 1.4 is the following

  Corollary 1.1 *Let φ be C^1 on $\mathbb{X} \subset \mathbb{R}$ and $\bar{x} \in \mathbb{X}$ be an interior steady state of φ. Then, the following holds:*

  (i) If $|\varphi'(\bar{x})| < 1$, then $\bar{x}$ is locally asymptotically stable.

  (ii) If $|\varphi'(\bar{x})| > 1$, then $\bar{x}$ is unstable.

- Remarks:
  - a fixed point $\bar{x}$ for which $|\varphi'(\bar{x})| \neq 1$ is called *hyperbolic*
  - only stability properties of hyperbolic fixed points can be inferred from $\varphi'(\bar{x})$
  - in the non-hyperbolic case $|\varphi'(\bar{x})| = 1$, additional conditions must be checked.

1.8 Equilibrium dynamics in the OLG model

1.8.1 Existence and non-existence of steady states

- Know:
  - equilibrium dynamics in OLG model determined by map $\mathcal{K}$ defined in Section 1.6.5
  - $\mathcal{K}$ is C^1, strictly increasing, and bounded by $\mathcal{W}$
- But: Existence of a non-trivial steady state $\bar{k} > 0$ not guaranteed, may well have

$$\mathcal{K}(k) < k$$

  for all $k > 0$ ('impoverishment').
- Sufficient condition to exclude this and ensure existence of steady state $\bar{k} > 0$ is

$$\lim_{k \searrow 0} \mathcal{K}'(k) > 1.$$

- Monotonicity of $\mathcal{K}$:
  - implies monotonic convergence/divergence of all orbits
  - excludes cyclical behavior

1.8.2 Uniqueness and multiplicity of steady states

- Existence of a steady state does not imply uniqueness
- Examples discussed in class.
- For further discussion and explicit restrictions on fundamentals to obtain existence/uniqueness see Galor & Ryder (1989)

2 Decision problems with infinite time horizon

2.1 A prototype consumption investment problem

2.1.1 Decision setup

- Consider the same problem as in Section 1.2 but with $T = \infty$
- Planing horizon $\mathbb{T} := \{0, 1, 2, \ldots\}$ is now (countably) infinite
- Given variables:
  - sequence of non-capital incomes $e^\infty = (e_t)_{t\in\mathbb{T}} \geq 0$
  - sequence of capital returns $R^\infty = (R_t)_{t\in\mathbb{T}} \gg 0$
  - initial capital $\bar{s}_{-1}$ (to be restricted)
- Decision variables:
  - consumption plan $(c_t)_{t\in\mathbb{T}} \geq 0$
  - investment plan $(s_t)_{t\in\mathbb{T}}$

2.1.2 NPG-condition and intertemporal budget set

- As before, feasible plans satisfy budget equation

$$c_t + s_t \leq e_t + R_t s_{t-1} \tag{38}$$

for all $t \in \mathbb{T}$ where $s_{-1} = \bar{s}_{-1}$

- For $t > 0$, define and interpret

$$q_t := R_1^{-1} \dots R_t^{-1} = \prod_{n=1}^{t} R_n^{-1} \tag{39}$$

as price of time t consumption in units of time zero consumption [**why?**]

- Using (39), we also impose the No-Ponzi Game (NPG) condition

$$\lim_{t \to \infty} q_t s_t \geq 0. \tag{40}$$

- Interpretation of (40): All loans must ultimately be repaid!
- Feasible plans are defined by budget set:

$$\mathbb{B}(e^\infty, R^\infty, \bar{s}_{-1}) = \Big\{ (c_t, s_t)_{t \in \mathbb{T}} | \, c_t \geq 0, (38) \text{ holds for all } t \in \mathbb{T}, (40) \text{ holds} \Big\} \tag{41}$$

2.1.3 Preferences and decision problem

- Preferences over consumption plans $(c_t)_{t\in\mathbb{T}} \in \mathbb{R}_+^{\mathbb{T}}$ represented by utility function

$$U((c_t)_{t\in\mathbb{T}}) := \sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1. \tag{42}$$

- Decision problem:

$$\max\Big\{U((c_t)_{t\in\mathbb{T}})\Big|(c_t, s_t)_{t\in\mathbb{T}} \in \mathbb{B}(e^{\infty}, R^{\infty}, \bar{s}_{-1})\Big\}. \tag{43}$$

- Maintain Assumption 1.2 on utility and assume that $0 < \beta < 1$.
- Remark:
  - we will set aside problems with infinity by assuming that incomes $(e_t)_{t\in\mathbb{T}}$ and prices $(q_t)_{t\in\mathbb{T}}$ are 'well-behaved' such that $|U((c_t)_{t\in\mathbb{T}})| < \infty$ for all $(c_t, s_t)_{t\in\mathbb{T}} \in \mathbb{B}(e^{\infty}, R^{\infty}, \bar{s}_{-1})$
  - explicit conditions under wihch this holds can easily be formulated.

2.1.4 Lifetime budget constraint

- Remainder assumes that (38) holds with equality for all t (due to monotonic preferences)
- Use (38) (with equality) to recursively eliminate $(s_t)_{t\in\mathbb{T}}$ from decision
- Using (39), obtain for all $T \geq 1$:

$$q_T s_T = \sum_{t=0}^{T} q_t(e_t - c_t) + R_0 \bar{s}_{-1}. \tag{44}$$

- Taking the limit $T \to \infty$ and using (40) gives

$$\underbrace{\sum_{t=0}^{\infty} q_t c_t}_{\text{lifetime consumption expenditure}} \leq \underbrace{\sum_{t=0}^{\infty} q_t e_t}_{\text{lifetime income}} + \underbrace{R_0 \bar{s}_{-1}}_{\text{initial capital income}}. \tag{45}$$

- We will assume the solvency condition that lifetime wealth is positive and finite:

$$0 < M := \sum_{t=0}^{\infty} q_t e_t + R_0 \bar{s}_{-1} < \infty. \tag{46}$$

- Can now write (44) as

$$\sum_{t=0}^{\infty} q_t c_t \leq M. \tag{47}$$

to obtain lifetime budget set:

$$\tilde{\mathbb{B}}(q^{\infty}, M) := \Big\{(c_t)_{t\in\mathbb{T}} \big| \, (46) \text{ holds } \wedge \, c_t \geq 0 \, \forall t \in \mathbb{T}\Big\}. \tag{48}$$

where $q^{\infty} := (q_t)_{t\in\mathbb{T}}$.

- Can state (43) in the following equivalent form:

$$\max\Big\{U((c_t)_{t\in\mathbb{T}}) \Big| (c_t)_{t\in\mathbb{T}} \in \tilde{\mathbb{B}}(q^{\infty}, M)\Big\}. \tag{49}$$

- Given a solution $(c_t^*)_{t\in\mathbb{T}}$ to (49), can easily recover optimal savings $(s_t^*)_{t\in\mathbb{T}}$ from (44).

2.2 Solving the problem by Lagrangian methods

- Under conditions satisfied here, Lagrangian techniques also applicable in infinite-dimensional cases (cf. Dechert (1982) or Le Van & Saglamb (2004))

- Define the Lagrangian function

$$\mathscr{L}((c_t, \mu_t)_{t\in\mathbb{T}}, \lambda) := \sum_{t\in\mathbb{T}} \Big[\beta^t u(c_t) + \mu_t c_t\Big] + \lambda M - \lambda \sum_{t\in\mathbb{T}} q_t c_t$$

- As in the finite-dimensional case, $(c_t^*)_{t\in\mathbb{T}} \geq 0$ solves (49) if there exist non-negative Lagrangian multipliers $(\mu_t^*)_{t\in\mathbb{T}}$ and $\lambda^* \geq 0$ such that the first order conditions (FOCs):

$$\frac{\partial \mathscr{L}}{\partial c_t}((c_t, \mu_t)_{t\in\mathbb{T}}, \lambda) = \beta^t u'(c_t) + \mu_t - \lambda q_t = 0 \quad \forall t \in \mathbb{T} \tag{50}$$

  the complementary slackness conditions (CSCs):

$$\mu_t c_t = \lambda M - \lambda \sum_{t\in\mathbb{T}} q_t c_t = 0 \tag{51}$$

  and (47) are satisfied for all t.

- Arguments analogous to Section 1.2 yield $\mu_t^* = 0$ for all $t \in \mathbb{T}$ and $\lambda^* > 0$.

Proposition 2.1 *Let Assumption 1.2 and the solvency condition (46) hold. Then, any sequence* $(c_t^*)_{t\in\mathbb{T}}$ *which solves*

$$\beta\frac{u'(c_{t+1})}{u'(c_t)} = \frac{q_{t+1}}{q_t} = \frac{1}{R_{t+1}} \quad \forall t \in \mathbb{T} \tag{52}$$

$$\sum_{t\in\mathbb{T}} q_t c_t = \sum_{t\in\mathbb{T}} q_t e_t + R_0 \bar{s}_{-1} \tag{53}$$

is a solution to (49).

- Equation (53) implies that optimal investment sequence $(s_t^*)_{t\in\mathbb{T}}$ defined by (44) satisfies

$$\lim_{t\to\infty} q_t s_t^* = 0 \tag{54}$$

- We will call (54) the transversality condition (TVC)
- As the optimal solution satisfies (52) for all t, $q_t = \beta^t u'(c_t^*)/u'(c_0^*)$
- As $u'(c_0^*)$ is just a constant, the TVC can equivalently be written as

$$\lim_{t\to\infty} \beta^t u'(c_t^*) s_t^* = 0 \tag{55}$$

- As (53) and (54) are equivalent, solution to original problem (43) can be characterized as:

Proposition 2.2 *Let Assumption 1.2 and the solvency condition (46) hold. Then, any sequence* $(c_t^*, s_t^*)_{t\in\mathbb{T}}$ *which satisfies*

$$\beta\frac{u'(c_{t+1})}{u'(c_t)} = \frac{q_{t+1}}{q_t} = \frac{1}{R_{t+1}} \tag{56}$$

$$c_t + s_t = e_t + R_t s_{t-1} \tag{57}$$

for all $t \in \mathbb{T}$ *(where* $s_{-1} = \bar{s}_{-1}$*) as well as the TVC (54) is a solution to (43).*

- Remark:
  - Although not obvious, the TVC is in fact a restriction on initial consumption c_0!
  - Will get back to this in Section 2.6

2.3 Solving the problem by variational methods

- Can also employ the following variational argument to obtain (52)
- Consider a variation of the optimal decision $(c_t^*)_{t\in\mathbb{T}}$ in period t_0 setting

$$\begin{aligned}\tilde{c}_{t_0} &= c_{t_0}^* - \delta \\ \tilde{c}_{t_0+1} &= c_{t_0+1}^* + \delta R_{t_0+1}\end{aligned}$$

  where δ is a small number.

- All other choices remain unchanged, can write utility as function H of δ:

$$H(\delta) := \beta^{t_0} u(c_{t_0}^* - \delta) + \beta^{t_0+1} u(c_{t_0+1}^* + \delta R_{t_0+1}) + \sum_{t\in\mathbb{T}\setminus\{t_0,t_0+1\}} \beta^t u(c_t^*). \qquad (58)$$

- Since $(c_t^*)_{t\in\mathbb{T}}$ is optimal, H must be maximal for $\delta = 0$. This implies $H'(0) = 0$ which gives (52) for all t.

2.4 An equilibrium framework: The neoclassical growth model

- Embed previous problem into a dynamic macro-model with
  - a consumption sector consisting of N identical infinite-lived consumers
  - a production sector represented by a single firm

2.4.1 Consumption sector

- Each consumer:
  - planes over infinitely many future periods $\mathbb{T} = \{0, 1, 2, \ldots\}$ as in Section 2.1
  - supplies one unit of labor to the labor market in each period t
  - consumes c_t, invests s_t which becomes capital k_{t+1} in $t+1$
  - capital earns return R_t, labor the wage w_t in t
- As consumers are identical, so are the decisions they take!

2.4.2 Production sector

- Production sector identical to Section 1.6.3, impose Assumption 1.3 on f
- Given labor $L_t = N$ and capital $K_t = Nk_t$, factor prices w_t and R_t determined by (33)

2.4.3 Consumer behavior

- Given her initial capital $k_0 > 0$, consumer chooses non-negative consumption-capital sequence (c_t, k_{t+1}) subject to budget constraint

$$k_{t+1} + c_t = w_t + k_t R_t, \quad \forall t \in \mathbb{T} \tag{59}$$

  to maximize utility $U((c_t)_{t\in\mathbb{T}})$ given by (42)

- Decision problem special case of (43) (where $e_t = w_t$, $s_t = k_{t+1}$, $\bar{s}_{-1} = k_0 > 0$)
- By Proposition 2.2, optimal decision characterized by (59), the Euler equations

$$\beta \frac{u'(c_{t+1})}{u'(c_t)} = R_{t+1}^{-1} \quad \forall t \in \mathbb{T} \tag{60}$$

  and the transversality condition

$$\lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = 0. \tag{61}$$

2.4.4 Equilibrium

- Economy is summarized by the list $\mathscr{E}_{NC} = \langle u, \beta, f \rangle$

Definition 2.1 *Given $k_0 > 0$, an equilibrium of $\mathscr{E}_{NC}$ is an allocation $(c_t^e, k_{t+1}^e)_{t\geq 0}$ and a price sequence $(w_t^e, R_t^e)_{t\geq 0}$ which satisfy (33), (59), (60), and (61)*

2.5 Equilibrium dynamics in state space form

- Equilibrium allocation solves the following implicit equations for all $t \geq 0$ by:

$$\begin{aligned} k_{t+1} + c_t - f(k_t) &= 0 && \text{(62a)} \\ \beta u'(c_{t+1}) f'(k_{t+1}) - u'(c_t) &= 0. && \text{(62b)} \end{aligned}$$

- As u' is strictly decreasing and therefore, invertible, obtain explicit form of (62):

$$\begin{aligned} k_{t+1} &= \varphi_k(k_t, c_t) := f(k_t) - c_t && \text{(63a)} \\ c_{t+1} &= \varphi_c(k_t, c_t) := u'^{-1}\left(\frac{u'(c_t)}{\beta f'(f(k_t) - c_t)}\right). && \text{(63b)} \end{aligned}$$

- System (63) represents the equilibrium dynamics in **state space form**
- $\varphi = (\varphi_k, \varphi_c)$ defined on

$$\mathbb{X} = \left\{(k, c) \in \mathbb{R}^2_{++} | c < f(k)\right\} \tag{64}$$

  but will see that $(\varphi, \mathbb{X})$ is not a dynamical system!

- Interior steady states $(\bar{k}, \bar{c}) \gg 0$ of (63) solve $f'(\bar{k}) = 1/\beta$ and $\bar{c} = f(\bar{k}) - \bar{k}$
- **Show:** Under Assumptions 1.2 and 1.3, there exists a unique interior steady state $(\bar{k}, \bar{c})$

2.6 A geometric interpretation of the transversality condition

2.6.1 An unstable steady state which is saddle-path stable

- System (63) offers nice geometric characterization of the TVC
- Will show this for special case where $u(c) = \log c$[1] and $f(k) = k^{\alpha}$, $0 < \alpha < 1$
- All qualitative insights extend to general case!
- Under previous parametrization, mapping φ in (63) reads:

$$\varphi_k(k, c) = k^{\alpha} - c \tag{65a}$$
$$\varphi_c(k, c) = \alpha\beta c \left(k^{\alpha} - c\right)^{\alpha - 1}. \tag{65b}$$

- Show (see Galor (2007) for additional details!):
  - system (65) has unique steady state $\bar{x} := (\bar{k}, \bar{c})$
  - Jacobian $D\varphi(\bar{x})$ has (real) Eigenvalues $|\lambda_1| < 1 < |\lambda_2|$
  - thus, $\bar{x}$ is unstable, in fact, *saddle-path stable* (cf. Problem 2.2 (iii) on PS 2!)
  - but: convergence towards $\bar{x}$ on lower-dimensional subset $\mathbb{M} \subset \mathbb{X}$ (stable manifold)

[1] Recall Section 1.5 and the remarks given there!

2.6.2 The stable manifold

- To determine $\mathbb{M}$, define $z_t := c_t/k_t^\alpha$ which evolves as

$$z_{t+1} = \psi(z_t) := \alpha\beta \frac{z_t}{1 - z_t} \tag{66}$$

where ψ is defined on $]0,1[$, but $(\psi,]0,1[)$ is not a dynamical system!

- Dynamic properties of ψ [**show!**]:
  - ψ has unique non-trivial steady state $\bar{z} = 1 - \alpha\beta > 0$ which is unstable
  - for $z_0 < \bar{z}$, $\lim_{t\to\infty} \psi^t(z_0) = 0$ which implies $\lim_{t\to\infty} c_t = 0$ whenever $c_0 < \bar{z}k_0^\alpha$
  - for $z_0 > \bar{z}$, $\psi^{t_0}(z_0) > 1$ for finite $t_0 \geq 1$ which implies $k_{t_0+1} < 0$ whenever $c_0 > \bar{z}k_0^\alpha$
- Sequence $(k_t, c_t)_{t\geq 0}$ generated by (65) well-defined and does not diverge iff $c_0 = \bar{z}k_0^\alpha$
- Conclude that:
  - stable manifold is $\mathbb{M} = \{(k, c) \in \mathbb{R}^2_{++} | c = (1 - \alpha\beta)k^\alpha\}$
  - equilibrium allocation satisfies $(k_t^e, c_t^e) \in \mathbb{M}$ for all $t \geq 0$ and $\lim_{t\to\infty}(k_t^e, c_t^e) = (\bar{k}, \bar{c})$
  - condition $c_0 = \bar{z}k_0^\alpha$ is equivalent to TVC (61) (cf. picture in class)
  - $\mathscr{E}_{NC}$ has a unique equilibrium which also holds in the general case!

2.7 A planning problem

- There is an alternative foundation for the equilibrium dynamics in the neoclassical model
- Consider a benevolent social planner who maximizes consumer utility by choosing a feasible allocation.

Definition 2.2 *Given $k_0 > 0$, a feasible allocation is a sequence $(c_t, k_{t+1})_{t\geq 0}$ which satisfies $c_t \geq 0$, $k_{t+1} \geq 0$ for all $t \geq 0$ as well as the resource constraint*

$$k_{t+1} + c_t \leq f(k_t). \tag{67}$$

The set of feasible allocations is denoted $\mathbb{A}(k_0)$.

- The planning problem reads:

$$\max_{(c_t,k_{t+1})_{t\in\mathbb{T}}} \left\{ U((c_t)_{t\in\mathbb{T}}) \,\middle|\, (c_t, k_{t+1})_{t\in\mathbb{T}} \in \mathbb{A}(k_0) \right\} \tag{68}$$

- Will see that the solution to (68) coincides with the equilibrium allocation!

2.8 Solving the planning problem by Lagrangian methods

- To solve (68), can use a variational argument as in Section 2.3 (or a Lagrangian approach as in Section 2.2) to obtain the conditions (62) for all $t \geq 0$.
- Will prove that if the solution to (62) satisfies the TVC (61), it solves the SPP (68).

Proposition 2.3 *Under Assumptions 1.2 and 1.3, the following holds for all* $k_0 > 0$*:*

(i) Any sequence $(c_t^*, k_{t+1}^*)_{t \geq 0}$ *which solves (62) for all* $t \geq 0$ *and (61) is a solution to (68).*

(ii) Any solution to (68) is unique.

Proof: Done in class.

- Previous result implies that equilibrium allocation is unique which in turn implies a unique equilibrium (**why?**)
- Solutions to (68) are precisely the Pareto-optimal allocations of $\mathscr{E}_{NC}$ (**why?**)
- Thus,there is an equivalence between *equilibrium* and *Pareto optimal allocations*
- Economically, this is a consequence of the first and second **Welfare Theorems**!
- Note that Proposition 2.3 does not, in general, deliver an existence result! This will be obtained next using recursive methods similar to Section 1.3.

2.9 Solving the planning problem by recursive methods

2.9.1 The Bellman equation

- Assume that f satisfies Assumption 1.3 and u the (weaker) Assumption 1.1 and $0 < \beta < 1$
- For each $k > 0$, define

$$V(k) := \sup_{(c_t, k_{t+1})_{t \in \mathbb{T}}} \Big\{ U((c_t)_{t \in \mathbb{T}}) \,\Big|\, (c_t, k_{t+1})_{t \in \mathbb{T}} \in \mathbb{A}(k) \Big\} \tag{69}$$

- We assume that the economy $\mathscr{E}$ is well-behaved such that $V(k) < \infty$ for all $k > 0$
- Principle of optimality implies that V solves functional equation (**Bellman-equation**):

$$V(k) = \max_{c, k_+} \Big\{ u(c) + \beta V(k_+) \,\Big|\, c \geq 0, k_+ \geq 0, k_+ + c \leq f(k) \Big\} \tag{70}$$

  or, equivalently,

$$V(k) = \max_{k_+} \Big\{ u(f(k) - k_+) + \beta V(k_+) \,\Big|\, 0 \leq k_+ \leq f(k) \Big\} \tag{71}$$

- The solution V to (71) is called the **value function**

2.9.2 Existence of a solution to the Bellman equation

- For the following results, see Stokey & Lucas (1989) or Stachurski (2009)
- Following holds if u is bounded:
  - Bellman equation (70) has a unique solution $V : \mathbb{R}_+ \longrightarrow \mathbb{R}$ which is also bounded, strictly increasing, and strictly concave
  - V is a fixed point of an operator T which maps the space $\mathscr{C}(\mathbb{R}_+)$ of bounded continuous functions $G : \mathbb{R}_+ \longrightarrow \mathbb{R}$ into itself, i.e., $T : \mathscr{C}(\mathbb{R}_+) \longrightarrow \mathscr{C}(\mathbb{R}_+)$ and $TV = V$
  - T is a contraction on $\mathscr{C}(\mathbb{R}_+)$ which is a Banach space under the sup-norm.
  - By the *Contraction Mapping Theorem*, V is unique and $\lim_{n\to\infty} T^n G = V$ for all $G \in \mathscr{C}(\mathbb{R}_+)$ (where convergence is in the sup-norm)
  - This implies that V is continuous, increasing, and concave.
- Essentially same results hold if u not bounded but homogeneous of degree $\theta \leq 1$, cf. Alvarez & Stokey (1998) (as in Problems 1.1 and 3.1 which essentially have $\theta = 1 - \sigma$)

2.9.3 Policy function and an existence theorem

- Knowing V, can compute the **policy function** $\mathcal{K} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$,

$$\mathcal{K}(k) = \arg\max_{k_+} \Big\{ u(f(k) - k_+) + \beta V(k_+) \,\Big|\, 0 \le k_+ \le f(k) \Big\}. \tag{72}$$

  which determines optimal capital formation k_{t+1} in t as a function of current capital k_t.

- Policy function $\mathcal{K}$ is continuous by the *Theorem of the Maximum* (cf. Stokey & Lucas (1989, Theorem 3.6, p.62)) and satisfies $0 < \mathcal{K} < f$.

- Given $\mathcal{K}$, define the consumption function $\mathcal{C} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$,

$$\mathcal{C}(k) := f(k) - \mathcal{K}(k) \tag{73}$$

  which determines optimal consumption c_t in period t as a function of current capital k_t.

- Following is the main result of this section.

Proposition 2.4 *Let V be the solution to (70) and define $\mathcal{K}$ and $\mathcal{C}$ as in (72) and (73). Then, the sequence $(k^*_{t+1}, c^*_t)_{t\ge 0}$ defined recursively for $t \ge 0$ as:*

$$k^*_{t+1} = \mathcal{K}(k^*_t) \tag{74a}$$

$$c^*_t = \mathcal{C}(k^*_t) \tag{74b}$$

where $k^*_0 = k_0$ is a solution to (68).

2.9.4 Differentiability of the value function V

- Let u satisfy the stronger Assumption 1.2 and f satisfy Assumption 1.3 and $f(0) = 0$. These restrictions imply an interior solution to (71) for all $k > 0$.

- To recover first order conditions (62) of SPP (68) from (71) V would need to be differentiable to apply the envelope theorem as in Section 1.4.2

- As argued above, V obtains as the limit of continuous functions (under the sup norm) and is, therefore, continuous but need not be differentiable!

- However, can apply the *Beneviste-Scheinkman Theorem* to prove that V is differentiable (cf. Stokey & Lucas (1989), Theorems 4.10 and 4.11, pp.84/85) and satisfies for all $k > 0$:

$$V'(k) = u'(f(k) - \mathcal{K}(k))f'(k) \tag{75}$$

- In this case, $k_+ = \mathcal{K}(k)$ is determined by the first order conditions

$$u'(f(k) - k_+) + \beta V'(k_+) = 0. \tag{76}$$

- Combining (75) and (76) and setting $\mathcal{C}(k) := f(k) - \mathcal{K}(k)$ gives for all $k > 0$:

$$u'(\mathcal{C}(k)) + \beta f'(k_+)u'(\mathcal{C}(k_+)) = 0 \tag{77}$$

  which implies Euler equation (62b) when (77) is evaluated at optimal sequence $(k_t^*)_{t\geq 0}$ generated by (74a).

2.10 A beautiful result that connects the two approaches

- Following is the nexus between the

  1. *Lagrangian approach* from Section 2.2 which gave us the dynamics (63) in state space form
  2. *recursive approach* from Section 2.9 which gave us the policy functions (72) and (73)

- Let $\bar{x} = (\bar{k}, \bar{c})$ be the unique interior steady state of $\varphi = (\varphi_k, \varphi_c)$ from (63) defined on $\mathbb{X}$ as in (64). Define the stable manifold

$$\mathbb{M} := \Big\{ x = (k, c) \in \mathbb{X} \,|\, \varphi^t(k, c) \in \mathbb{X} \ \forall t \geq 0 \wedge \lim_{t \to \infty} \varphi^t(k, c) = (\bar{k}, \bar{c}) \Big\}. \tag{78}$$

- Let $\mathcal{K}$ be the policy defined in (72) and $\mathcal{C}$ the consumption function (73). Define its graph

$$\text{graph}(\mathcal{C}) := \Big\{ (k, c) \in \mathbb{X} \,|\, c = \mathcal{C}(k) \Big\}. \tag{79}$$

- Then, we have:

$$\mathbb{M} = \text{graph}(\mathcal{C}). \tag{80}$$

Mathematical Methods, Part 1: Applied Intertemporal Optimization

Part II

Winter term 2015/16

Lecturer: Marten Hillebrand
Teaching assistant: Van Hoang Khieu

Part II
Stochastic Models in Discrete Time

3 Setting up a random environment

- In a stochastic world, all quantities take the form of random variables
- We will first review some basic concepts from probability theory required for our purposes
- Following structure is very condensed. You are strongly encouraged to consult Stachurski (2009) for a more in-depth treatment.
- In the sequel we work with infinite discrete time periods $\mathbb{T} = \{0, 1, 2, \ldots\}$.
- If A is any set, 2^A or $\mathrm{Pow}(A)$ denotes the power set, i.e., the class of all subsets of A.

3.1 Basic concepts from probability theory

3.1.1 Probability spaces and random variables

- Randomness in our model enters via an exogenous stochastic process $(\theta_t)_{t\geq 0}$, i.e., a sequence of random variables with values in $\Theta \subset \mathbb{R}^N$, $N \geq 1$.

- All these random variables live on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where:

  - Ω is the *sample space* which represents all possible states of the world
  - $\mathscr{F} \subset \text{Pow}(\Omega)$ is a collection of subsets of Ω that form a σ*-algebra*, i.e., (i) $\Omega \in \mathscr{F}$, (ii) $A \in \mathscr{F}$ implies $A^c := \Omega \backslash A \in \mathscr{F}$ and (iii) $(A_n)_{n\geq 0}$, $A_n \in \mathscr{F}\ \forall n$ implies $\cup_{n=0}^{\infty} A_n \in \mathscr{F}$.
  - $\mathbb{P} : \mathscr{F} \longrightarrow [0,1]$ is a probability measure, i.e., a countably additive function satisfying $\mathbb{P}(\Omega) = 1$ that assigns probabilities $\mathbb{P}(A)$ to each measurable subset $A \in \mathscr{F}$ of Ω

- Θ is endowed with some σ-algebra $\mathscr{A} \subset \text{Pow}(\Theta)$ to become a measurable space $(\Theta, \mathscr{A})$

- Since $\Theta \subset \mathbb{R}^N$ is a topological space, we can (and typically do) choose for $\mathscr{A}$ the Borel-σ algebra $\mathscr{B}(\Theta)$ which is the smallest σ-algebra containing the topology

- For each $t \in \mathbb{T}$, the mapping $\theta_t : \Omega \longrightarrow \Theta$ is $\mathscr{F} - \mathscr{B}(\Theta)$ measurable, i.e., for all $B \in \mathscr{B}(\Theta)$, $\theta_t^{-1}(B) := \{\omega \in \Omega | \theta_t(\omega) \in B\} \subset \Omega$ is an element of $\mathscr{F}$.

3.1.2 Probability and distributions of random variables

- For each $t \in \mathbb{T}$, can construct probability distribution/measure μ_t of random variable θ_t:
  - given a set $B \in \mathscr{B}(\Theta)$, $\mu_t(B)$ is the probability that $\theta_t \in B$
  - straightforward to construct μ_t by defining the *image measure*

$$\mu_t(B) = \mathbb{P}\left(\theta_t^{-1}(B)\right) = \mathbb{P}\left(\{\omega \in \Omega | \theta_t(\omega) \in B\}\right) \tag{81}$$

  - mapping $\mu_t : \mathscr{B}(\Theta) \longrightarrow [0,1]$ is indeed a probability measure on $(\Theta, \mathscr{B}(\Theta))$ and called the probability distribution of θ_t
  - if $\Theta = \mathbb{R}$, there is a one-to one correspondence between distribution μ_t and the *distribution function* $F_t(b) := \mu_t(]-\infty, b])$, $b \in \mathbb{R}$. Similar result holds if $N > 1$.
- Analogously, construct joint distribution $\mu_{\mathbb{I}}$ of random variables $\theta_{\mathbb{I}} := (\theta_t)_{t \in \mathbb{I}}$ for any $\mathbb{I} \subset \mathbb{T}$
- Further, can infer the distributions of random variables defined by measurable functions
  - $f : \Theta \longrightarrow \mathbb{X} \subset \mathbb{R}^M$ of θ_t
  - $f : \Theta^{\mathbb{I}} \longrightarrow \mathbb{X} \subset \mathbb{R}^M$ of $\theta_{\mathbb{I}}$

  with values in the measurable space $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$

3.2 Constructing the underlying probability space

- Previous result: Given $(\Omega, \mathscr{F}, \mathbb{P})$ and measurable mappings $(\theta_t)_t$, can compute probability distributions of all random variables $(\theta_t)_{t\in\mathbb{I}}$, $\mathbb{I} \subset \mathbb{T}$ and of all measurable functions of these random variables

- Can also reverse the previous construction:
  - specify distributions/dependence structure of the random variables $(\theta_t)_{t\in\mathbb{T}}$
  - construct an underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ consistent with this.

3.2.1 Example 1: Independent random variables

- Suppose we want $(\theta_t)_{t\geq 0}$ to consist of independent random variables wit values in Θ each having a desired probability distribution $\mu : \mathscr{B}(\Theta) \longrightarrow [0,1]$, say, a normal distribution.

- In this case, define:
  - $\Omega = \Theta^{\mathbb{T}}$ (the set of sequences with values in Θ)
  - $\mathscr{F} = \mathscr{B}(\Omega)$ (the product σ-algebra generated by measurable rectangles or, equivalently, the Borel σ-algebra when Ω is endowed with the product topology)
  - $\mathbb{P} = \mu^{\mathbb{T}}$ (the product measure which satisfies $\mu^{\mathbb{T}}(\Omega \times \ldots \times \Omega \times A \times B \times \Omega \times \ldots) = \mu(A) \cdot \mu(B)$ for any $A, B \in \mathscr{B}(\Theta)$)

3.2.2 Example 2: Correlated random variables

- Suppose we want $(\theta_t)_{t\in\mathbb{T}}$ to follow an auto-regressive structure of the form

$$\theta_t = M\theta_{t-1} + \varepsilon_t, \quad t \geq 1, \tag{82}$$

  where $M \in \mathbb{R}^{N\times N}$ and $(\varepsilon_t)_{t\geq 1}$ consists of i.i.d. random variables with values in $\mathcal{E} \subset \mathbb{R}^N$ and distribution μ_ε which are independent of θ_0 which has distribution μ_0.

- In this case, can also construct $(\Omega, \mathscr{F}, \mathbb{P})$ by defining $\Omega = \Theta\times\mathcal{E}^{\mathbb{N}}$, $\mathscr{F} = \mathscr{B}(\Omega)$, $\mathbb{P} = \mu_0\otimes\mu_\varepsilon^{\mathbb{N}}$.

- Noting that $\theta_t = A^t\theta_0 + \sum_{n=0}^{t-1} M^n\varepsilon_{t-n}$ we can compute μ_t for each $t > 0$ via (81)

- For later reference, note that (121) defines a *transition probability*, i.e., a mapping $Q : \Theta \times \mathscr{B}(\Theta) \longrightarrow [0,1]$ such that $Q(\theta, A)$ is the probability that $\theta_{t+1} \in A$ given that $\theta_t = \theta$

- For all $\theta \in \Theta$ and $A \in \mathscr{B}(\Theta)$, Q can explicitly be constructed as

$$Q(\theta, A) = \mu_\varepsilon\{\varepsilon \in \mathcal{E} | M\theta + \varepsilon \in A\} \tag{83}$$

- The distributions $(\mu_t)_{t\in\mathbb{T}}$ can then be computed recursively for $t \geq 1$ as

$$\mu_t(B) = \int_\Theta Q(\theta, B)\mu_{t-1}(d\theta). \tag{84}$$

  for each $B \in \mathscr{B}(\Theta)$.

3.3 Filtration and conditional expectation

- Let $(\theta_t)_{t\geq 0}$ be the exogenous stochastic process on $(\Omega, \mathscr{F}, \mathbb{P})$ defined previously.
- In our equilibrium framework derived below, all endogenous variables will take the form of random variables $(X_t)_{t\geq 0}$ with values in $\mathbb{X} \subset \mathbb{R}^M$ which depend on the exogenous process.
- We generally take the notation $(X_t)_{t\geq 0}$ to mean that X_t is observable in period t, i.e., can only depend on exogenous random variables θ_n, $n \leq t$.
- To impose this restriction formally, define a *filtration* $(\mathscr{F}_t)_{t\geq 0}$ where $\mathscr{F}_t \subset \mathscr{F}$ is the smallest σ-algebra such that each θ_n, $0 \leq n \leq t$ is $\mathscr{F}_t$-$\mathscr{B}(\Theta)$ measurable.
- Process $(X_t)_{t\in\mathbb{T}}$ is said to be *adapted* (to $(\mathscr{F}_t)_{t\geq 0}$) if each X_t is $\mathscr{F}_t - \mathscr{B}(\mathbb{X})$ measurable. This captures exactly the idea that X_t can depend only on random variables θ_n, $n \leq t$
- Specifically, if $(X_t)_{t\in\mathbb{T}}$ is adapted and $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathscr{F}_t]$ is the expectations operator conditional on observations up to time t, $\mathbb{E}_t[X_n] = X_n$ for all t and $n \leq t$.
- If X_t has distribution $\mu_{X_t} : \mathscr{B}(\mathbb{X}) \longrightarrow [0, 1]$ and is integrable, its unconditional expectation is defined as

$$\mathbb{E}[X_t] := \int_\Omega X_t(\omega)\mathbb{P}(d\omega) = \int_\mathbb{X} x\mu_{X_t}(dx). \tag{85}$$

4 Stochastic decision problems with finite horizon

4.1 A stochastic OLG model

- Consider a stochastic version of the OLG model from Section 1.6 similar to Wang (1993):
  - all assumptions on population structure, labor supply, etc. remain the same
  - but: production side modified to incorporate random production shocks
- We continue to denote equilibrium variables as $(X_t)_{t \geq 0}$ but these are now adapted stochastic processes rather than just sequences.
- All equalities and inequalities involving random variables are assumed to hold $\mathbb{P}$-almost surely without explicit notice.

4.1.1 Production side

- Suppose that production in period t is subject to multiplicative shock $\theta_t \in \Theta \subset \mathbb{R}_{++}$:

$$Y_t = \theta_t F(K_t, L_t) = \theta_t L_t f(k_t) \tag{86}$$

- Production shocks $(\theta_t)_{t \geq 0}$ consists of independent random variables with distribution μ and values in $\Theta = [\theta_{\min}, \theta_{\max}] \subset \mathbb{R}_{++}$.
- Thus, we can chose the underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ exactly as outlined in Section 3.2.1
- In each period t, the firm takes the current shock in period t as given and decides on demand for capital and labor.
- Continue to impose Assumption 1.3 on f and define $k_t = \frac{K_t}{L_t}$ for all $t \in \mathbb{T}$ as before.
- The fist order conditions then determine equilibrium factor prices as:

$$w_t = \mathcal{W}(k_t, \theta_t) := \theta_t[f(k_t) - k_t f'(k_t)] \tag{87a}$$
$$R_t = \mathcal{R}(k_t, \theta_t) := \theta_t f'(k_t) \tag{87b}$$

4.1.2 A stochastic two-period decision problem

- Consider decision problem of a young consumer in period $t \geq 0$:
  - consumer observes her current labor income $w_t > 0$ (which is a real number)
  - capital return R_{t+1} treated as random variable with values in $[R_{\min}, R_{\max}] \subset \mathbb{R}_{++}$
  - knowing the underlying probabilistic structure, consumer computes correct conditional expectation $\mathbb{E}_t[\cdot]$ of next period's random variables
- Any investment decision $s_t \in [0, w_t]$ (which is a number!) determines lifetime consumption

$$c_t^y = w_t - s_t \tag{88a}$$

$$c_{t+1}^o = R_{t+1} s_t \tag{88b}$$

  where $c_t^y \geq 0$ is a number and c_{t+1}^o a random variable with values in $[s_t R_{\min}, s_t R_{\max}]$.

- Preferences over alternative random variables (c_t^y, c_{t+1}^o)possess an expected utility representation with von-Nemann Morgenstern utility $U(c^y, c^o) = u(c^y) + \beta u(c^o)$
- Decision problem reads:

$$\max_s \left\{ u(w_t - s) + \beta \mathbb{E}_t \left[u(s R_{t+1}) \right] \mid 0 \leq s \leq w_t \right\} \tag{89}$$

- Define consumer's objective function $U_t :]0, w_t[\longrightarrow \mathbb{R}$,

$$U_t(s) := u(w_t - s) + \beta \mathbb{E}_t \left[u(sR_{t+1}) \right] \tag{90}$$

- Imposing Assumption 1.2 on utility u, we obtain the following result:

  Lemma 4.1 *Under Assumption 1.2, the following holds:*

  (i) U_t in (90) is C^2, strictly concave, and $\lim_{s \searrow 0} U_t'(s) = -\lim_{s \nearrow w_t} U_t'(s) = -\infty$

  (ii) Problem (89) has a unique interior solution s_t^* determined by

$$u'(w_t - s) = \beta \mathbb{E}_t \left[R_{t+1} u'(sR_{t+1}) \right] \tag{91}$$

- **Hint:** When proving this result, exploit that in the present case, differentiation can be interchanged with the expectations operator

4.1.3 Deriving the equilibrium equations

- Aggregate investment made at time t determines next periods's capital stock $K_{t+1} = Ns_t$
- Defining $k_{t+1} = K_{t+1}/N$, (91) can be written as:

$$u'(w_t - k_{t+1}) = \beta \mathbb{E}_t \left[R_{t+1} u'(k_{t+1} R_{t+1}) \right] \tag{92}$$

- Observations:
  - by (87b), next period's capital return determined by $R_{t+1} = \theta_{t+1} f'(k_{t+1})$
  - uncertainty in R_{t+1} completely due uncertainty about shock which has distribution μ independent of any other realizations at time t
  - this permits (92) to be written as:

$$\begin{aligned} u'(w_t - k_{t+1}) &= \beta \mathbb{E}_\mu \left[\mathcal{R}(k_{t+1}, \cdot) u'(k_{t+1} \mathcal{R}(k_{t+1}, \cdot)) \right] \qquad (93) \\ &= \beta \int_\Theta \mathcal{R}(k_{t+1}, \theta) u'(k_{t+1} \mathcal{R}(k_{t+1}, \theta)) \mu(d\theta). \end{aligned}$$

- Consumption of both generations in t satisfies:

$$c_t^y = w_t - k_{t+1} \tag{94a}$$

$$c_t^o = R_t k_t. \tag{94b}$$

4.1.4 Equilibrium

- Stochastic OLG economy is summarized by the list $\mathscr{E}_{SOLG} = \langle u, \beta, f, \mu \rangle$
- Following definition of equilibrium is straightforward generalization of deterministic case.

Definition 4.1 *Given $k_0 > 0$, an equilibrium of $\mathscr{E}_{SOLG}$ consists of adapted stochastic processes of prices $(w_t^e, R_t^e)_{t \geq 0}$ and an allocation $(k_{t+1}^e, c_t^{y,e}, c_t^{o,e})_{t \geq 0}$ satisfying (87), (93), and (94) for all $t \geq 0$.*

- Questions as in the deterministic case:
  - existence of equilibrium?
  - uniqueness of equilibrium?
  - dynamic behavior of equilibrium?
- To answer them, will again derive recursive structure of equilibrium.

4.1.5 Recursive structure of equilibrium

- Following ideas exactly analogous to deterministic case studied in Section 1.6.5
- Given $k > 0$ and $\theta \in \Theta$, define for each $0 < k_+ < \mathcal{W}(k, \theta)$ the function

$$H(k_+; k, \theta) := u'(\mathcal{W}(k, \theta) - k_+) - \beta \mathbb{E}_\mu \left[\mathcal{R}(k_+, \cdot)\, u'(k_+ \mathcal{R}(k_+, \cdot)) \right]. \tag{95}$$

- Equilibrium process $(k^e_{t+1})_{t \geq 0}$ solves $H(k_{t+1}; k_t, \theta_t) = 0$ for all $t \geq 0$ and determines all other equilibrium variables
- Following auxiliary result can be proved exactly as in the deterministic case:

Lemma 4.2 *Under Assumptions 1.2 and 1.3, the following holds:*

(i) The function $H(\cdot; k, \theta)$ defined in (95) has at least one zero for all $k > 0$ and $\theta \in \Theta$.

(ii) If, in addition either (a) or (b) of Assumption hold, this zero is unique.

- Lemma 4.2 allows us to state the following main result:

Proposition 4.1 *Under Assumptions 1.2 and 1.3, the following holds for all $k_0 > 0$:*

(i) Economy $\mathscr{E}_{\mathrm{SOLG}}$ has at least one equilibrium.

(ii) If, in addition, either (a) or (b) of Assumption 1.4 hold, this equilibrium is unique.

- Observations:
  - additional restrictions ensure existence of a map $\mathcal{K} : \mathbb{R}_{++} \times \Theta \longrightarrow \mathbb{R}_{++}$ which determines the unique solution $k_+ = \mathcal{K}(k, \theta)$ to (34) for each $k > 0$ and $\theta \in \Theta$
  - by the implicit function theorem, $\mathcal{K}$ is C^1, strictly increasing, and satisfies

$$0 < \mathcal{K}(k, \theta) < \mathcal{W}(k, \theta) < f(k, \theta). \tag{96}$$

  - unique equilibrium process $(k^e_{t+1})_{t \geq 0}$ determined recursively by

$$k^e_{t+1} = \mathcal{K}(k^e_t, \theta_t). \tag{97}$$

- To study equilibrium dynamics, need some basic concepts from stochastic dynamical systems theory

5 Stochastic dynamical systems in discrete time

5.1 Stochastic dynamical systems and stability

- For more details, the reader is again referred to Stachurski (2009).
- Assume that endogenous state dynamics take the form $F : \mathbb{X} \times \Theta \longrightarrow \mathbb{X}$

  $$x_{t+1} = F(x_t, \theta_t) \tag{98}$$

  where we now restrict attention to case where $\mathbb{X} = \mathbb{R}_+$
- Also assume that exogenous process is i.i.d. with distribution μ_θ and values in $\Theta = [\theta_{\min}, \theta_{\max}] \subset \mathbb{R}_{++}$
- In the deterministic case, the state x_t in period t is a real number
- In the stochastic case, the state x_t in period t is a random variable which is completely described by its distribution $\mu_t : \mathscr{B}(\mathbb{X}) \longrightarrow [0, 1]$
- Thus, a steady state in the stochastic case is a distribution $\bar{\mu}$ (or a random variable $\bar{x}$ which has this distribution) which remains invariant under (98).
- Thus, to compute a stochastic steady state of (98), we need to derive how the sequence of distributions $(\mu_t)_{t \geq 0}$ of the random variables $(x_t)_{t \geq 0}$ evolve over time

5.2 Markov operator

- Suppose x_0 has distribution μ_0, what is the distribution μ_t of x_t for any $t \geq 1$?
- As in Section 3.2.2, note that (98) defines a *transition probability*, i.e., a mapping $Q : \mathbb{X} \times \mathscr{B}(\mathbb{X}) \longrightarrow [0,1]$ such that $Q(x, A)$ is the probability that $x_{t+1} \in A$ given that $x_t = x$
- For all $x \in \mathbb{X}$ and $A \in \mathscr{B}(\mathbb{X})$, Q can explicitly be constructed as

$$Q(x, A) = \mu_\theta\{\theta \in \Theta | F(x, \theta) \in A\} \tag{99}$$

- The distributions $(\mu_t)_{t \in \mathbb{T}}$ can then be computed recursively for $t \geq 1$ as

$$\mu_t(B) = \int_{\mathbb{X}} Q(x, B)\mu_{t-1}(dx). \tag{100}$$

  for each $B \in \mathscr{B}(\mathbb{X})$.

- Let $\mathscr{M}(\mathbb{X})$ denote the class of probability measures on $\mathscr{B}(\mathbb{X})$
- Then, can define an operator $T : \mathscr{M}(\mathbb{X}) \longrightarrow \mathscr{M}(\mathbb{X})$ which associates with any $\mu \in \mathscr{M}(\mathbb{X})$ the new probability measure $T\mu$ defined for each $B \in \mathscr{B}(\mathbb{X})$ as

$$T\mu(B) = \int_{\mathbb{X}} Q(x, B)\mu(dx). \tag{101}$$

5.3 Stochastic steady states

- The concept of an invariant distribution is now straightforward:

  Definition 5.1 *An steady state of the stochastic dynamical system (98) is a probability distribution $\bar{\mu} \in \mathscr{M}(\mathbb{X})$ which is a fixed point of T, i.e., $T\bar{\mu} = \bar{\mu}$.*

- The stochastic analog of a steady state is therefore an invariant probability distribution

- Large literature which studies existence of invariant distributions for Markov operators

- Notion of stability requires $\lim_{t\to\infty} T^t \mu_0 = \bar{\mu}$ where the limiting operation requires a suitable notion of convergence of measures (most results on stability use the concept of weak convergence, see Stokey & Lucas (1989)).

- Very general conditions for existence/uniqueness/stablity of invariant distributions if F resp. T has certain monotonicity properties in Kamihigashi & Stachurski (2014)

- There is also a theory of *Random Dynamical Systems* due to Arnold (1998) which defines the concept of a random fixed point.

- See Schenk-Hoppé & Schmalfuss (2001) for an economic application of this theory and how it relates to the previous concepts

5.4 Equilibrium dynamics in the stochastic OLG model

5.4.1 Stable sets

- Consider existence of stochastic steady states/invariant distributions of economy $\mathscr{E}_{\text{SOLG}}$
- Existence of stochastic steady states follows from the existence of *stable sets*:

Definition 5.2 *A stable set of (97) is an interval* $[k_{\min}, k_{\max}] \subset \mathbb{R}_{++}$ *such that:*

(i) $\mathcal{K}(k_{\min}, \theta_{\min}) = k_{\min}$

(ii) $\mathcal{K}(k_{\max}, \theta_{\max}) = k_{\max}$

(iii) $\mathcal{K}(k, \theta_{\min}) < k < \mathcal{K}(k, \theta_{\max})$ *for all* $k \in [k_{\min}, k_{\max}]$

- Existence of a stable set non-trivial steady state $\bar{k} > 0$ not guaranteed, fails if

$$\mathcal{K}(k, \theta_{\max}) < k \tag{102}$$

  for all $k > 0$ (impoverishment).

- Sufficient condition to exclude this and ensure existence is

$$\lim_{k \searrow 0} \mathcal{K}'(k, \theta_{\min}) > 1. \tag{103}$$

5.4.2 Existence of a stochastic steady state

- Following existence result due to Wang (1993):

  Proposition 5.1 (Wang (1993)) *If the equilibrium map $\mathcal{K}$ from (97) satisfies condition (103), there exists at least one stochastic steady state/invariant probability distribution.*

- Uniqueness of a stable sets not guaranteed, same multiplicity problem as in the deterministic case.

- Uniqueness obtains, however, if for all $\theta \in \Theta$, $\mathcal{K}(\cdot, \theta)$ has a unique fixed point.

- This is a special case of the more general concept of a *stable fixed point configuration* (cf. Brock & Mirman (1972)). Essentially, this requires that the largest fixed point of $\mathcal{K}(\cdot, \theta_{\min})$ be smaller than the smallest fixed point of $\mathcal{K}(\cdot, \theta_{\max})$ (cf. the illustrations provided in class).

- Much more general existence results on stochastic steady states that also hold for a much larger class of OLG economies can be found, e.g., in Morand & Reffett (2007) and McGovern, Morand & Reffett (2013).

6 Stochastic decision problems with infinite horizon

6.1 A prototype decision problem

- Consider the problem as in Section 2.2 with $\mathbb{T} := \{0, 1, 2, \ldots\}$ but now with uncertainty
- In particular, we now:
  - abstract from loans by requiring $s_t \geq 0$.
  - include the consumer's labor-leisure choice $h_t \in [0, \bar{h}]$ which determines labor supply
- Remainder normalizes maximum labor to $\bar{h} = 1$.

6.1.1 Decision setup

- Given variables:
  - adapted stochastic process of wages $w^\infty = (w_t)_{t\in\mathbb{T}}$
  - adapted stochastic process of capital returns $R^\infty = (R_t)_{t\in\mathbb{T}}$
  - initial capital $\bar{s}_{-1} \geq 0$
- Decision variables:
  - consumption plan: adapted stochastic process $(c_t)_{t\in\mathbb{T}} \geq 0$, $c_t \geq 0 \ \forall t$
  - investment plan: adapted stochastic process $(s_t)_{t\in\mathbb{T}}$, $s_t \geq 0 \ \forall t$
  - labor supply plan: adapted stochastic process $(h_t)_{t\in\mathbb{T}}$, $0 \leq h_t \leq \bar{h} \ \forall t$

6.1.2 Intertemporal budget set

- Feasible plans must satisfy period budget equation

$$c_t + s_t \leq w_t h_t + R_t s_{t-1} \tag{104}$$

for all $t \in \mathbb{T}$ where $s_{-1} = \bar{s}_{-1}$

- Feasible plans are defined by budget set:

$$\mathbb{B}(w^\infty, R^\infty, \bar{s}_{-1}) = \Big\{ (c_t, h_t, s_t)_{t \in \mathbb{T}} |\, c_t \geq 0, 0 \leq h_t \leq 1, s_t \geq 0, (104) \text{ holds for all } t \in \mathbb{T} \Big\} \tag{105}$$

6.1.3 Preferences and decision problem

- Utility in period t now depends on consumption $c_t \geq 0$ and leisure $0 \leq h_t \leq 1$ and given by utility function

$$u : \mathbb{R}_+ \times [0,] \longrightarrow \mathbb{R}, \ (c, h) \mapsto u(c, h) \tag{106}$$

- Preferences over consumption-labor processes $(c_t, h_t)_{t\in\mathbb{T}}$ represented by utility function

$$U((c_t, h_t)_{t\in\mathbb{T}}) := \mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t u(c_t, h_t)\right], \quad 0 < \beta < 1. \tag{107}$$

- Decision problem:

$$\max\Big\{U((c_t, h_t)_{t\in\mathbb{T}})\Big|(c_t, h_t, s_t)_{t\in\mathbb{T}} \in \mathbb{B}(e^\infty, R^\infty, \bar{s}_{-1})\Big\}. \tag{108}$$

Assumption 6.1 *The utility function u in (106) is continuous, strictly concave and C^2 on the interior of its domain with partial derivatives satisfying*

$$\partial_{cc}u < 0 < \partial_c u \qquad \text{and} \qquad \lim_{c\searrow 0} \partial_c u(c, h) = \infty \tag{109a}$$

$$\partial_{hh}u < 0 < -\partial_h u \qquad \text{and} \qquad \lim_{h\nearrow 1} \partial_h u(c, h) = -\infty. \tag{109b}$$

6.2 Solving the decision problem

- Following derivations impose Assumption 6.1
- Then, any solution to (108) will be interior, i.e., $c_t^* > 0$ and $0 < h_t^* < 1$ due to (109)
- Can again use a variational argument to obtain following equations which characterize solution
- For each $t \geq 0$ and conditional on $\mathscr{F}_t$, solution to (108) must satisfy the intratemporal optimality condition

$$-\frac{\partial_h u(c_t, h_t)}{\partial_c u(c_t, h_t)} = w_t \tag{110}$$

  and the intertemporal optimality condition (Euler equation)

$$\mathbb{E}_t\left[R_{t+1}\frac{\beta\partial_c u(c_{t+1}, h_{t+1})}{\partial_c u(c_t, h_t)}\right] = 1. \tag{111}$$

- Further, for all $t \geq 0$, the budget equality

$$c_t + s_t = w_t h_t + R_t s_{t-1} \tag{112}$$

holds and the stochastic transversality condition (STVC)

$$\lim_{T\to\infty} \mathbb{E}_0\left[s_T \prod_{t=0}^{T} R_t^{-1}\right] = 0. \tag{113}$$

- Remarks:
  - that any process $(c_t^*, s_t^*, h_t^*)_{t\in\mathbb{T}} \in \mathbb{B}(e^\infty, R^\infty, \bar{s}_{-1})$ satisfying (110), (111), (112) for all $t \in \mathbb{T}$ as well as (113) is indeed a solution to (108) can be proved along the lines of the proof of Proposition 2.3 done in class (exploiting the law of iterated expectations!).
  - one can also show by using the same arguments as in the proof of Proposition 2.3 that the solution to (108) is $\mathbb{P}$-a.s. unique.
  - we could also - somewhat mechanically - have used a Lagrangian approach to obtain the previous conditions, but it is not quite clear how derivatives conditional on $\mathscr{F}_t$ should be interpreted.

6.3 An equilibrium framework: The RBC model

- Consider a stochastic version of the neoclassical growth model from Section 2.4 with endogenous labor supply:
  - production side modified to incorporate random production shocks
  - consumer side modified to include labor-leisure choice, decision problem solved under uncertainty as in Section 6.1
  - unless stated otherwise, all other assumptions remain the same as in Section 2.4
- We continue to denote equilibrium variables as $(X_t)_{t\geq 0}$ but these are now adapted stochastic processes rather than just sequences.
- All equalities and inequalities involving random variables are assumed to hold $\mathbb{P}$-almost surely without explicit notice.

6.3.1 Consumer side

- As in deterministic case, N identical consumers who each
  - plan over infinitely many future periods $\mathbb{T} = \{0, 1, 2, \ldots\}$
  - consume c_t and invest s_t in period t
  - supply h_t units of labor in period t, now determined endogenously
  - capital earns return R_t, labor the wage w_t in t
- Decision problem exactly as in Section 6.1
- As consumers are identical, so are the decisions they take!
- At the aggregate level, factor supply in period $t \geq 0$ given by

$$L_t = N h_t \tag{114}$$

$$K_t = N s_{t-1} \tag{115}$$

and capital per capita $k_t := K_t/N$ evolves as

$$k_t = s_{t-1}, \quad t \geq 1. \tag{116}$$

- Optimal decision satisfies the conditions:

$$c_t + k_{t+1} = w_t h_t + R_t k_t \tag{117a}$$

$$-\frac{\partial_h u(c_t, h_t)}{\partial_c u(c_t, h_t)} = w_t \tag{117b}$$

$$\mathbb{E}_t \left[R_{t+1} \frac{\beta \partial_c u(c_{t+1}, h_{t+1})}{\partial_c u(c_t, h_t)} \right] = 1 \tag{117c}$$

and the stochastic transversality condition (STVC)

$$\lim_{T \to \infty} \mathbb{E}_0 \left[k_{T+1} \prod_{t=0}^{T} R_t^{-1} \right] = 0. \tag{118}$$

6.3.2 Production side

- Suppose that (net) production in period t is subject to multiplicative shock $\theta_t \in \Theta \subset \mathbb{R}_{++}$ such that total output (including non-depreciated capital) is given by:

$$Y_t = e^{\theta_t} F(K_t, L_t) + (1-\delta)K_t = N\left[e^{\theta_t} h_t f(k_t/h_t) + (1-\delta)k_t\right] \tag{119}$$

- Continue to assume linear homogeneity of F and impose Assumption 1.3 on f.

- Using (114) and (115) and linear homogeneity, per capita output $y_t := Y_t/N$ given by

$$y_t = e^{\theta_t} F(k_t, h_t) + (1-\delta)k_t = e^{\theta_t} h_t f(k_t/h_t) + (1-\delta)k_t \tag{120}$$

- Production shocks $(\theta_t)_{t\geq 0}$ follow an AR(1)-process of the form

$$\theta_t = \rho\theta_{t-1} + \varepsilon_t \tag{121}$$

  where $0 \leq \rho < 1$ and $(\varepsilon_t)_{t\geq 0}$ consist of i.i.d. random variables with distribution μ_ε

- Thus, we can chose the underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and construct transition probability Q induced by (121) exactly as outlined in Section 3.2.2

- In each period t, the firm takes the current shock in period t as given and decides on demand for capital and labor.

- The fist order conditions then determine equilibrium factor prices as:

$$\begin{aligned} w_t = \mathcal{W}(k_t, h_t, \theta_t) &:= e^{\theta_t} \partial_h F(k_t, h_t) && (122a) \\ &= e^{\theta_t} \left[f(k_t/h_t) - k_t/h_t f'(k_t/h_t) \right] \\ R_t = \mathcal{R}(k_t, h_t, \theta_t) &:= e^{\theta_t} \partial_k F(k_t, h_t) + (1 - \delta) && (122b) \\ &= e^{\theta_t} f'(k_t/h_t) + 1 - \delta \end{aligned}$$

- Remark:
  - in the deterministic case, non-depreciated capital was included in F resp. f which had the interpretation of a gross production function
  - here, we interpret F resp. f as a net production function and must, therefore, explicitly keep track of non-depreciated capital
  - the reason is that only net production output is affected by the shock.

6.4 Equilibrium

- Economy is summarized by the list $\mathscr{E}_{RBC} = \langle u, \beta, N, f, Q \rangle$ plus initial conditions $k_0 > 0$ and $\theta_0 \in \Theta$

Definition 6.1 *Given $k_0 > 0$ and $\theta_0 \in \Theta$, an equilibrium of $\mathscr{E}_{RBC}$ is an allocation $(c_t^e, h_t^e, k_{t+1}^e)_{t\geq 0}$ and a price sequence $(w_t^e, R_t^e)_{t\geq 0}$ which satisfy (117) and (122) for all $t \geq 0$ and (118).*

- Can again use an equivalent planning problem to determine the (unique) equilibrium allocation

- Equilibrium prices then follow directly from (122) for all $t \geq 0$.

6.5 A stochastic planning problem

- Consider a benevolent social planner who maximizes consumer utility by choosing a feasible allocation.

  Definition 6.2 *Given $k_0 > 0$ and $\theta_0 \in \Theta$, a feasible allocation is an adapted stochastic process $(c_t, h_t, k_{t+1})_{t\geq 0}$ which satisfies $c_t \geq 0$, $0 \leq h_t \leq 1$, $k_{t+1} \geq 0$ for all $t \geq 0$ as well as the resource constraint*

  $$k_{t+1} + c_t \leq e^{\theta_t} F(k_t, h_t) + (1-\delta)k_t. \tag{123}$$

  The set of feasible allocations is denoted $\mathbb{A}(k_0, \theta_0)$.

- The planning problem reads:

$$\max_{(c_t,h_t,k_{t+1})_{t\in\mathbb{T}}} \left\{ U((c_t, h_t)_{t\in\mathbb{T}}) \,\middle|\, (c_t, h_t, k_{t+1})_{t\in\mathbb{T}} \in \mathbb{A}(k_0, \theta_0) \right\} \tag{124}$$

- As in the deterministic case, can compute the equations that characterize the solution to (124)

- Can show that these coincide with the equilibrium equations derived above.

- Thus, the solution to (68) also constitutes an equilibrium allocation!

6.6 Solving the stochastic planning problem by recursive methods

6.6.1 The Bellman equation

- Motivation for the following approach is analogous to the deterministic case
- Basic idea: Exploit the recursive structure of SPP
- Assume that f satisfies Assumption 1.3 and u Assumption 6.1 and $0 < \beta < 1$
- For brevity, set

$$M(k, h, \theta) := e^{\theta} F(k, h) + (1 - \delta)k \tag{125}$$

- In the present stochastic setup, the Bellmann equation reads:

$$V(k, \theta) = \max_{k_+ \geq 0, 0 \leq h \leq 1} \left\{ u(M(k, h, \theta) - k_+, h) + \beta \int_{\Theta} V(k_+, \theta_+) Q(\theta, d\theta_+) \,|\, k_+ \leq M(k, h, \theta) \right\} \tag{126}$$

6.6.2 Policy function

- Having computed the value function V, suppose the maximizing solution (k_+^*, h^*) in (126) is well-defined and unique for each $(k,\theta) \in \mathbb{R}_{++} \times \Theta$
- Define the policy function $g = (g_k, g_h) : \mathbb{R}_{++} \times \Theta \longrightarrow \mathbb{R}_+ \times [0,1]$

$$g(k,\theta) \quad = \quad \arg \max_{k_+ \geq 0, 0 \leq h \leq 1} \left\{ u(M(k,h,\theta) - k_+, h) + \beta \int_\Theta V(k_+, \theta_+) Q(\theta, d\theta_+) \,|\, k_+ \leq M(k,h,\theta) \right\}$$

Lemma 6.1 *Let V be the unique solution to (126) and $g = (g_k, g_h)$ be defined as above. Then, for each (k_0, z_0) the sequence $\{c_t^\star, h_t^\star, k_{t+1}^\star\}_{t\geq 0}$ defined recursively as $k_0^\star = k_0$,*

$$\begin{aligned} k_{t+1}^\star &= g_k(k_t^\star, \theta_t) \\ h_t^\star &= g_h(k_t^\star, \theta_t) \\ c_t^\star &= M(k_t^\star, h_t^\star, \theta_t) - k_{t+1}^\star \end{aligned}$$

for all $t \geq 0$ is a solution to (124).

6.7 Equilibrium dynamics in the RBC model

- Consequences of previous results:
  - dynamics completely described by the endogenous state variable $\{k_t^\star\}_{t\geq 0}$ and the exogenous process $\{\theta_t\}_{t\geq 0}$
  - analogously to stochastic OLG model, can analyze dynamics, existence of invariant distributions, etc.
  - in general, mapping $g_k(\cdot;\theta)$ has a unique steady state $\bar{k}_\theta$ for all $\theta \in \Theta$

References

ALVAREZ, F. & N. L. STOKEY (1998): "Dynamic Programming with Homogeneous Functions", *Journal of Economic Theory*, 82, 167–189.

ARNOLD, L. (1998): *Random Dynamical Systems*. Springer-Verlag, Berlin a.o.

BROCK, W. A. & L. J. MIRMAN (1972): "Optimal Growth and Uncertainty: The Discounted Case", *Journal of Economic Theory*, 4, 479–513.

DECHERT, W. (1982): "Lagrange multipliers in infinite horizon discrete time optimal control models", *Journal of Mathematical Economics*, 9, 285–302.

GALOR, O. (2007): *Discrete Dynamical Systems*. Springer, Berlin, a.o.

GALOR, O. & H. E. RYDER (1989): "Existence, Uniqueness, and Stability of Equilibrium in an Overlapping–Generations Model with Productive Capital", *Journal of Economic Theory*, 49, 360–375.

HILLEBRAND, M. (2008): *Pension Systems, Demographic Change, and the Stock Market*, Bd. 610 of *Lecture Notes in Economics and Mathematical Systems*. Springer, Berlin, Heidelberg, New York.

Kamihigashi, T. & J. Stachurski (2014): “Stochastic Stability in Monotone Economies”, *Theoretical Economics*, 9, 383 – 407.

Le Van, C. & H. C. Saglamb (2004): “Optimal growth models and the Lagrange multiplier”, *Journal of Mathematical Economics*, 40, 393410.

Mas-Colell, A., M. D. Whinston & J. R. Green (1995): *Microeconomic Theory*. Oxford University Press, Oxford a.o.

McGovern, J., O. F. Morand & K. L. Reffett (2013): “Computing minimal state space recursive equilibrium in OLG models with stochastic production”, *Economic Theory*, 54, 623–674.

Morand, O. F. & K. L. Reffett (2007): “Stationary Markovian Equilibrium in Overlapping Generations Models with Stochastic Nonclassical Production and Markov Shocks”, *Journal of Mathematical Economics*, 43, 501–522.

Schenk-Hoppé, K. R. & B. Schmalfuss (2001): “Random fixed points in a stochastic Solow growth model.”, *Journal of Mathematical Economics*, 36, 19–30.

Stachurski, J. (2009): *Economic Dynamics: Theory and Computation*. MIT Press, Cambridge (Mass.) a.o.

Stokey, N. L. & R. E. J. Lucas (1989): *Recursive Methods in Economic Dynamics*. Harvard University Press, Cambridge (Mass.) a.o.

Wang, Y. (1993): "Stationary Equilibria in an Overlapping Generations Economy with Stochastic Production", *Journal of Economic Theory*, 61(2), 423–435.