Complete experiments in pseudoscalar meson photoproduction

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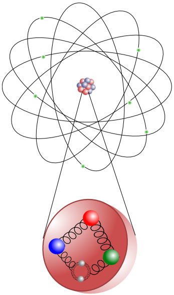
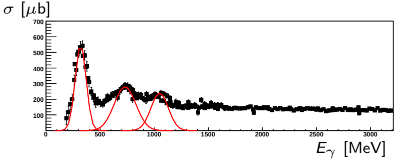
Motivation for photoproduction

- Spectroscopy: Excite the considered system energetically
  $\Rightarrow$ Learn about dynamics among the constituents

Atom

Baryon
(e.g. nucleon)





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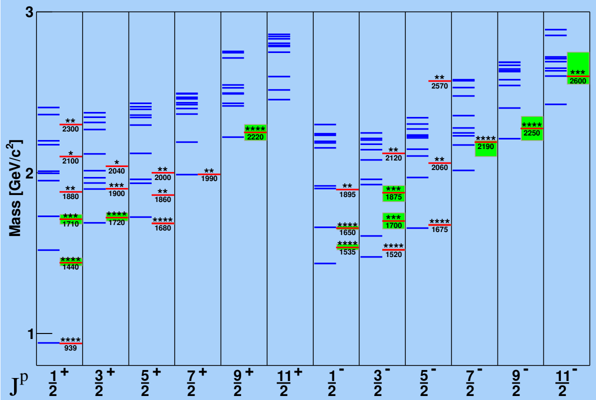
- Photoproduction data have already helped the identification of resonances missed in the πN scattering analyses



- N^* spectrum
- Predictions of the Bonn CQM on the left [Löring et al. (2001)]
- Resonances from [PDG (2014)] on the right

[Andrew Wilson]

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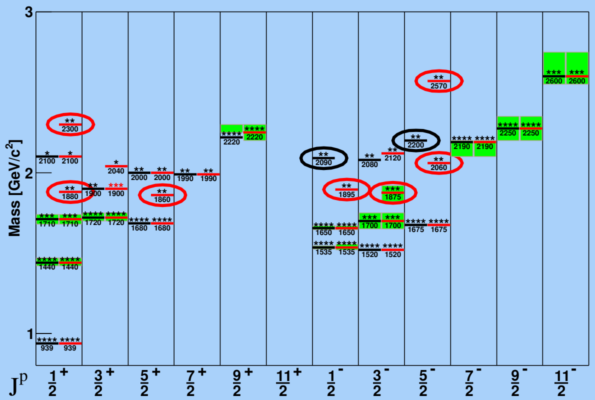
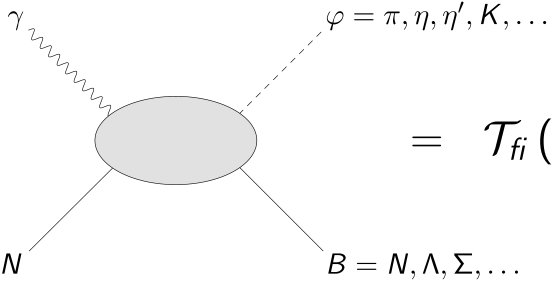


- N^* spectrum
- Resonances from [PDG (2010)] on the left vs. resonances from [PDG (2014)] on the right

[Andrew Wilson]

Photoproduction Amplitudes

The on-mass-shell amplitude for photoproduction $\gamma N \rightarrow \varphi B$ is



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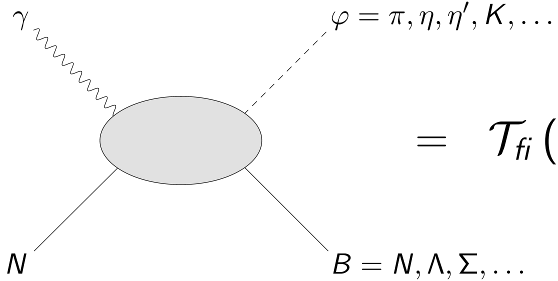

$$= \quad \mathcal{T}_{fi}\left(s, t\right)$$

It can be shown, using very general assumptions, that the production amplitude in the center of mass system (CMS) is:

$$\mathcal{T}_{fi}\left(s,t\right) = \mathcal{C}\chi^{\dagger}_{m_{s_f}}\left[i\vec{\sigma}\cdot\hat{\epsilon}F_1 + \vec{\sigma}\cdot\hat{q}\vec{\sigma}\cdot\left(\hat{k}\times\hat{\epsilon}\right)F_2 + i\vec{\sigma}\cdot\hat{k}\hat{q}\cdot\hat{\epsilon}F_3 + i\vec{\sigma}\cdot\hat{q}\hat{q}\cdot\hat{\epsilon}F_4\right]\chi_{m_{s_i}}$$

[Chew, Goldberger, Low and Nambu (1957)]

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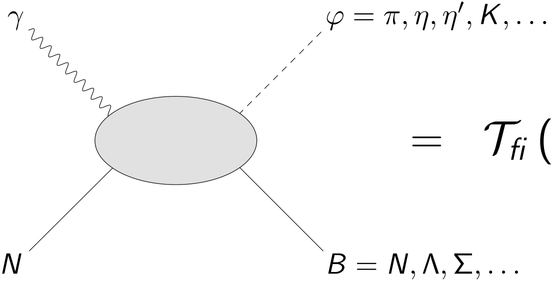

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[Chew, Goldberger, Low and Nambu (1957)]

$\rightarrow$ Process is fully described by 4 complex CGLN amplitudes $F_i\left(W, \theta\right)$

Polarization Observables I

Problem: 4 complex amplitudes $F_i\left(W,\theta\right) \equiv 8$ real numbers
$\Rightarrow$ 1 observable $\left(\frac{d\sigma}{d\Omega}\right)_0$ insufficient to determine the amplitudes!

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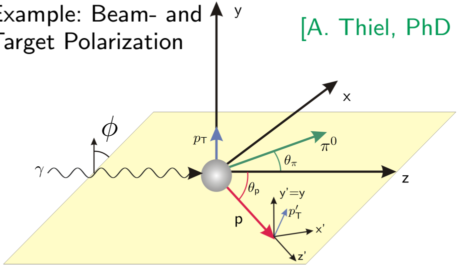
Example: Beam- and Target Polarization

[A. Thiel, PhD (2012)]



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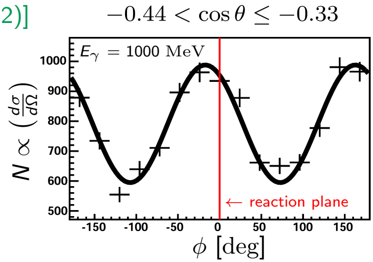

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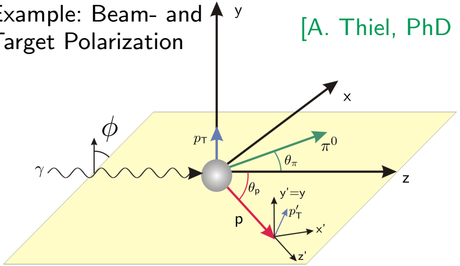

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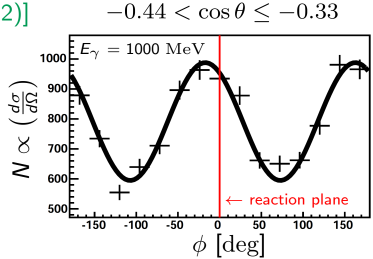

The observables Σ and G appear as amplitudes of the ϕ-modulations

$$\left(\frac{d\sigma}{d\Omega}\right)(\theta,\phi) = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(1 - \epsilon_L \Sigma \cos(2\phi) + \epsilon_L P_z^T G \sin(2\phi)\right).$$

E.g.: Σ is an asymmetry between polarization states $(\perp,0,0)$ & $(\parallel,0,0)$:

$$\Sigma = \frac{1}{2\left(\frac{d\sigma}{d\Omega}\right)_0} \left[\left(\frac{d\sigma}{d\Omega}\right)^{(\perp,0,0)} - \left(\frac{d\sigma}{d\Omega}\right)^{(\parallel,0,0)}\right].$$

Polarization Observables II

Generic definition of an observable

$$\Omega = \frac{\beta}{\sigma_0} \left[\left(\frac{d\sigma}{d\Omega} \right)^{(B_1, T_1, R_1)} - \left(\frac{d\sigma}{d\Omega} \right)^{(B_2, T_2, R_2)} \right]$$

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- In total, 16 non-redundant observables

$$\Omega^{\alpha} \left(W, \theta \right) = \frac{1}{2\sigma_0} \sum_{i,j} F_i^* \hat{A}_{ij}^{\alpha} F_j, \quad \alpha = 1, \ldots, 16$$

can be defined, involving Beam-, Target- and Recoil Polarization.

Beam		Target			Recoil			Target + Recoil			
	-	-	-	-	x'	y'	z'	x'	x'	z'	z'
	-	x	y	z	-	-	-	x	z	x	z
unpolarized	$\left(\frac{d\sigma}{d\Omega}\right)_0$		T			P		$T_{x'}$	$L_{x'}$	$T_{z'}$	$L_{z'}$
linear	Σ	H	P	G	$O_{x'}$	T	$O_{z'}$				
circular		F		E	$C_{x'}$		$C_{z'}$				

Multipole expansion I

- Expansion of amplitudes into angular momentum eigenstates

Non rel. QM / Spinless scattering

- 1 amplitude $f\left(W,\theta\right)$
- Partial wave expansion:

$$f\left(W,\theta\right)=\sum_{\ell=0}^{\infty}\left(2\ell+1\right)f_{\ell}\left(W\right)P_{\ell}\left(\cos\theta\right)$$

$P_{\ell}\left(\cos\left(\theta\right)\right)$: Legendre polynomials

$f_{\ell}\left(W\right)$: Partial wave amplitudes

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$f_{\ell}(W)$: Partial wave amplitudes

Photoproduction

- 4 amplitudes $F_i(W,\theta)$
- Partial wave expansion:

$$F_1(W,\theta) = \sum_{\ell=0}^{\infty} \Big\{ \left[\ell M_{\ell+} + E_{\ell+}\right] P_{\ell+1}^{'}(\cos(\theta)) + \left[(\ell+1) M_{\ell-} + E_{\ell-}\right] P_{\ell-1}^{'}(\cos(\theta)) \Big\}$$

$$F_2(W,\theta) = \sum_{\ell=1}^{\infty} \left[(\ell+1) M_{\ell+} + \ell M_{\ell-}\right] P_{\ell}^{'}(\cos(\theta))$$

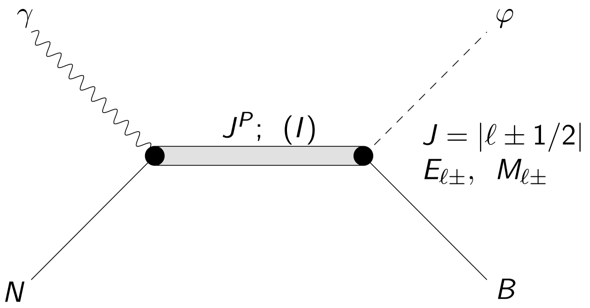
$$F_3(W,\theta) = \sum_{\ell=1}^{\infty} \Big\{ \left[E_{\ell+} - M_{\ell+}\right] P_{\ell+1}^{''}(\cos(\theta)) + \left[E_{\ell-} + M_{\ell-}\right] P_{\ell-1}^{''}(\cos(\theta)) \Big\}$$

$$F_4(W,\theta) = \sum_{\ell=2}^{\infty} \left[M_{\ell+} - E_{\ell+} - M_{\ell-} - E_{\ell-}\right] P_{\ell}^{''}(\cos(\theta))$$

$E_{\ell\pm}(W)$, $M_{\ell\pm}(W)$: multipoles

Multipole expansion II

- Correspondence between multipoles of certain quantum numbers and resonant intermediate states



ℓ: relative angular momentum

J: Total angular momentum

P: Parity

I: Isospin

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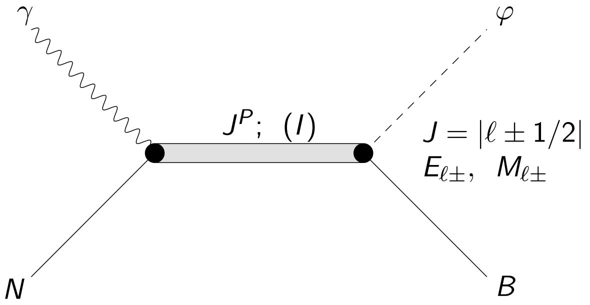

ℓ: relative angular momentum

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- Multipoles with definite isospin only for certain channels
  - E.g. photoproduction of pions
    - $\rightarrow$ $E^I_{\ell\pm}(W)$, $M^I_{\ell\pm}(W)$
    - $\rightarrow$ Isospin separation of nucleon and delta resonances

complete experiment problem

- Situation: 4 complex amplitudes (e.g. $F_i(W, \theta)$)

$$\updownarrow$$

16 real polarization observables $\check{\Omega}^{\alpha} = \frac{1}{2} \langle F | \hat{A}^{\alpha} | F \rangle$

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Solution: Theorem of Chiang & Tabakin

- 8 of 16 observables can yield a complete experiment
- All Group S observables $\left\{ \left(\frac{d\sigma}{d\Omega}\right)_0, \Sigma, \ T, \ P \right\}$ have to be measured
- The remaining 4 measurements must not belong to the same class (BT, BR or TR)
- No more than 2 observables are allowed to be picked from the same class

- Complete sets are tabulated [Chiang/Tabakin(1996)]

Algebraic calculation of amplitudes

- Observables have bilinear product form, e.g. for helicity amplitudes:
  $\check{\Omega}^{\alpha} = \frac{1}{2} \left< H \right| \Gamma^{\alpha} \left| H \right> = \frac{1}{2} \sum_{i,j} H_{i}^{\ast} \Gamma^{\alpha}_{ij} H_{j}$

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$\rightarrow$ Possibility to extract moduli $|H_i|$ and relative phases $\phi_{ij}^{H} = \phi_i^{H} - \phi_j^{H}$ from the formula

$$H_i^{*} H_j = \frac{1}{2} \sum_{\alpha} \left(\Gamma^{\alpha}_{ij} \right)^{*} \check{\Omega}^{\alpha}$$

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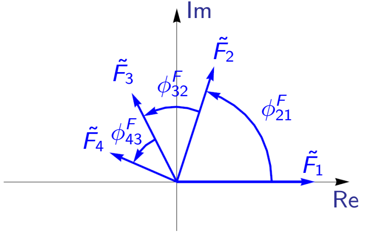
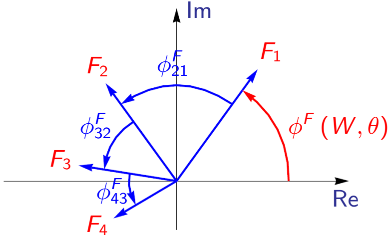
[Chiang/Tabakin(1996)]

- The expression can be generalized and applied to for example CGLN amplitudes $F_i \left(W, \theta \right)$

$\rightarrow$ Result:





Definition of complete experiments in a TPWA

Desirable for low-energy processes: Truncate the partial wave expansion of the full spin amplitudes at some finite ℓ_{max}, e.g.

$$F_1(W,\theta) = \sum_{\ell=0}^{\ell_{\mathrm{max}}} \Big\{ \left[\ell M_{\ell+} + E_{\ell+}\right] P^{'}_{\ell+1}\left(\cos\theta\right) + \left[\left(\ell+1\right) M_{\ell-} + E_{\ell-}\right] P^{'}_{\ell-1}\left(\cos\theta\right) \Big\},$$

and insert this truncated expansion into the polarization observables $\left\{\check{\Omega}^{\alpha}\left(W,\theta\right), \alpha = 1,\ldots,16\right\}$ of pseudoscalar meson photoproduction.

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Truncated Partial Wave Analysis

$$\check{\Omega}^{\alpha}(W,\theta) = \sin^{\beta_\alpha}\theta \left[a_0^{\alpha}(W) + a_1^{\alpha}(W)\cos\theta + a_2^{\alpha}(W)\cos^2\theta + \ldots\right]$$

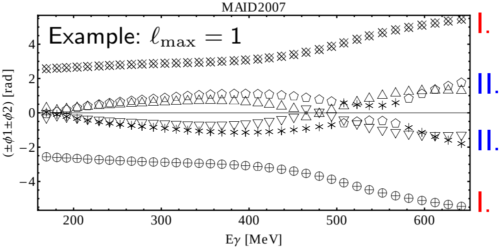
$$= \sin^{\beta_\alpha}\theta \sum_{k=0}^{2\ell_{\mathrm{max}}+\gamma_\alpha} a_k^{\alpha}(W)\cos^k\theta,$$

$$a_k^{\alpha}(W) = \langle\mathcal{M}(W)| \, C_k^{\alpha} \, |\mathcal{M}(W)\rangle, \ \ |\mathcal{M}(W)\rangle = \left(E_{\ell\pm}(W), M_{\ell\pm}(W)\right)^T.$$

$\rightarrow$ How many and which observables have to be measured in order to uniquely solve for the multipoles $\{E_{\ell\pm}(W), M_{\ell\pm}(W)\}$?

Complete sets of observables in a TPWA

Study of the theoretical discrete ambiguities of the group S observables $\left\{ \left(\frac{d\sigma}{d\Omega} \right)_0, \Sigma, P, T \right\}$ according to [A. S. Omelaenko (1981)] (see also [Wunderlich/Beck/Tiator (2014)])



Results of Ambiguity diagrams:

I. the double ambiguity can be predicted for all orders in ℓ_{max} and for all energies E_{γ}

II. accidential ambiguities may occur in each energy bin, but cannot be predicted $\rightarrow n = 4^{2\ell_{\mathrm{max}}} - 2$ (!!)

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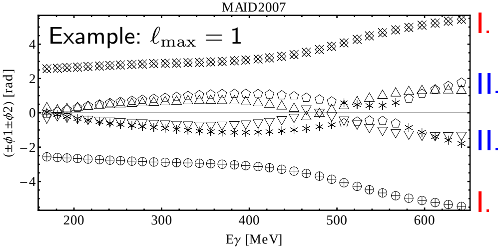

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$\rightarrow$ Double polarization observables capable of resolving the ambiguities:

$\mathcal{BT}$: $\{F, G\}$, $\mathcal{BR}$: $\{O_{x'}, O_{z'}, C_{x'}, C_{z'}\}$, $\mathcal{TR}$: $\{T_{x'}, T_{z'}, L_{x'}, L_{z'}\}$

$\rightarrow$ Examples of complete sets: $\underline{\{\sigma_0, \Sigma, T, P, F\}}$ or $\{\sigma_0, \Sigma, T, P, G\}$

$\rightarrow$ Can these statements be verified using numerical TPWA fits?

Details on the multipole Fit procedure I

Two step method:

1. Fit the angular distributions of observables, parametrized by

$$\check{\Omega}^{\alpha}\left(W,\theta\right) = \frac{q}{k} \sum_{k=\beta_{\alpha}}^{2\ell_{\mathrm{max}}+\beta_{\alpha}+\gamma_{\alpha}} \left(a_{L}\right)_{k}^{\alpha}\left(W\right) P_{k}^{\beta_{\alpha}}\left(\cos\theta\right)$$

$\Rightarrow$ Angular fit parameters $\left(a_{L}^{\mathrm{Fit}}\right)_{k}^{\alpha}$ & errors $\Delta\left(a_{L}^{\mathrm{Fit}}\right)_{k}^{\alpha}$

- Absorb $\sin^{\beta_{\alpha}}\theta$ factors into the fitting functions $P_{k}^{\beta_{\alpha}}\left(\cos\theta\right)$
- $P_{k}^{\beta_{\alpha}}\left(\cos\theta\right)$ have the advantage of being orthogonal for $\cos\theta \in [-1, 1]$

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2. Minimize the functional:

$$\Phi_{\mathcal{M}}\left(\mathcal{M}_{\ell}\right) = \frac{1}{N_{F.P.}-N_{V.M.}} \sum_{\alpha,k} \left(\frac{\left(\left(a_{L}^{\mathrm{Fit}}\right)_{k}^{\alpha} - \langle \mathcal{M}_{\ell} | \left(C_{L}\right)_{k}^{\alpha} | \mathcal{M}_{\ell} \rangle\right)}{\Delta\left(a_{L}^{\mathrm{Fit}}\right)_{k}^{\alpha}} \right)^{2}$$

using the MATHEMATICA method

$$\mathrm{FindMinimum}\left[\,\Phi_{\mathcal{M}}\left(\mathcal{M}_{\ell}\right), \left\{\left\{\mathrm{Re}\left[E_{0+}\right], \left(x_{1}\right)_{0}\right\}, \ldots, \left\{\mathrm{Im}\left[M_{\ell_{\mathrm{max}}-}\right], \left(y_{n}\right)_{0}\right\}\right\}\,\right]$$

and varying the real and imaginary parts of the (possibly phase constrained) multipoles in the fit.

Details on the multipole fit procedure II

Question: How to choose the start parameters $\left\{ \left(x_1 \right)_0 , \ldots , \left(y_n \right)_0 \right\}$?

Ansatz: Use the total cross section $\sigma \left(W \right)$. Example: $\ell \leq \ell_{\mathrm{max}} = 1$, phase constraint $\mathrm{Im} \left[\tilde{E}_{0+} \right] = 0$ & $\mathrm{Re} \left[\tilde{E}_{0+} \right] > 0$:

$$\sigma(W) \approx 4\pi \frac{q}{k} \Big(\mathrm{Re} \left[\tilde{E}_{0+} \right]^2 + 6 \, \mathrm{Re} \left[\tilde{E}_{1+} \right]^2 + 6 \, \mathrm{Im} \left[\tilde{E}_{1+} \right]^2 + 2 \, \mathrm{Re} \left[\tilde{M}_{1+} \right]^2 + 2 \, \mathrm{Im} \left[\tilde{M}_{1+} \right]^2 + \mathrm{Re} \left[\tilde{M}_{1-} \right]^2 + \mathrm{Im} \left[\tilde{M}_{1-} \right]^2 \Big)$$

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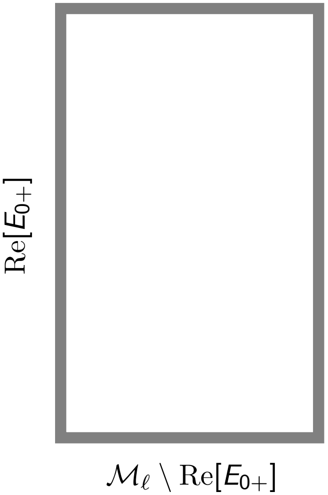
- $\sigma(W)$ constrains the intervals of the multipoles:

$$\mathrm{Re}\left[\tilde{E}_{0+}\right] \in \left[0, \sqrt{\tfrac{k}{q}\tfrac{\sigma(W)}{4\pi}}\right], \ldots, \mathrm{Im}\left[\tilde{M}_{1-}\right] \in \left[-\sqrt{\tfrac{k}{q}\tfrac{\sigma(W)}{4\pi}}, \sqrt{\tfrac{k}{q}\tfrac{\sigma(W)}{4\pi}}\right]$$

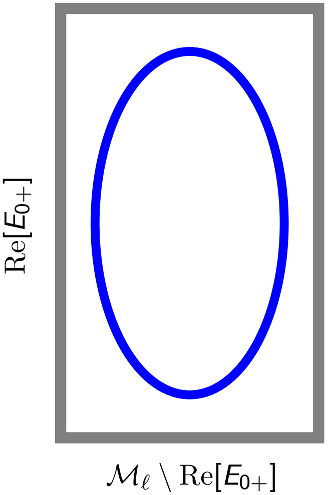
- The total cross section, being quadratic form in the multipoles, also defines an ellipsoid in the multipole space.

1. The total cross section $\sigma(W)$ constrains the $(8\ell_{\mathrm{max}} - 1)$-dimensional multipole space $\mathcal{M}_{\ell}$.



Generation of start values for FindMinimum

1. The total cross section $\sigma(W)$ constrains the $(8\ell_{\mathrm{max}}-1)$-dimensional multipole space $\mathcal{M}_{\ell}$.
2. $\sigma(W)$ defines an $(8\ell_{\mathrm{max}}-2)$-dimensional ellipsoid in $\mathcal{M}_{\ell}$.



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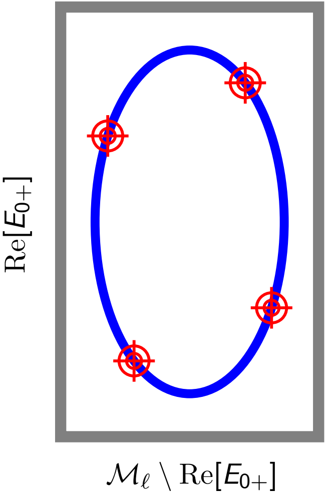
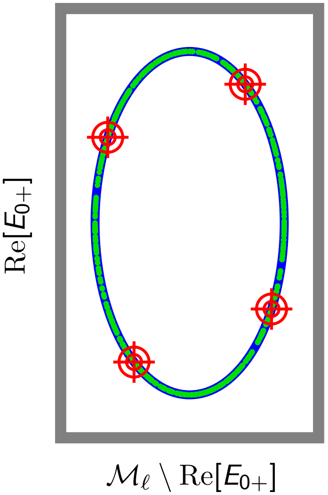
2. $\sigma(W)$ defines an $(8\ell_{\mathrm{max}} - 2)$-dimensional ellipsoid in $\mathcal{M}_{\ell}$.
3. Solutions to the TPWA problem lie on the ellipsoid defined by $\sigma(W)$.



Generation of start values for FindMinimum

1. The total cross section $\sigma(W)$ constrains the $(8\ell_{\mathrm{max}} - 1)$-dimensional multipole space $\mathcal{M}_{\ell}$.
2. $\sigma(W)$ defines an $(8\ell_{\mathrm{max}} - 2)$-dimensional ellipsoid in $\mathcal{M}_{\ell}$.
3. Solutions to the TPWA problem lie on the ellipsoid defined by $\sigma(W)$.
4. The start values for the $\mathrm{FindMinimum}$-Fit are chosen randomly on the $\sigma(W)$-ellipsoid.

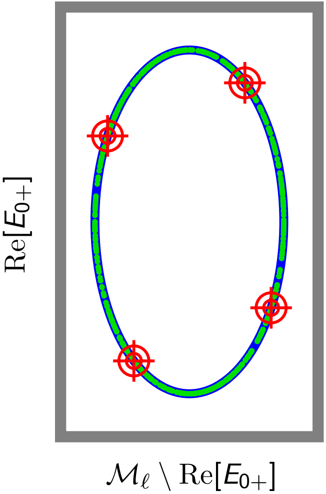
   $\Rightarrow$ Monte Carlo sampling of the multipole space.



Generation of start values for FindMinimum

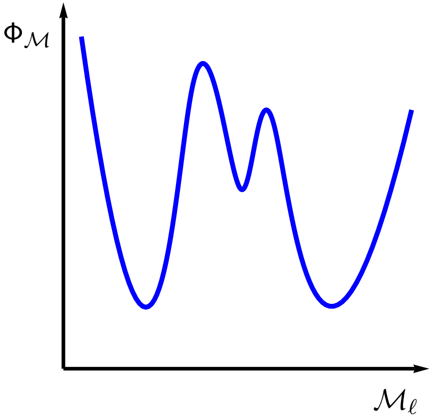
5. A FindMinimum-minimization is performed for each of the randomly generated start configurations.

   $\Rightarrow N_{MC} = \#$ of M.C. start configurations
   $= \#$ of (possibly redundant) solutions

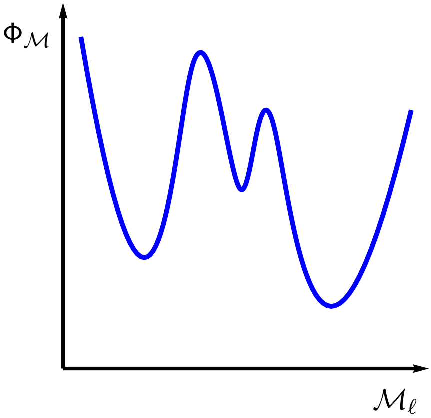


Cut selections for solution "data"

Mathematical ambiguity



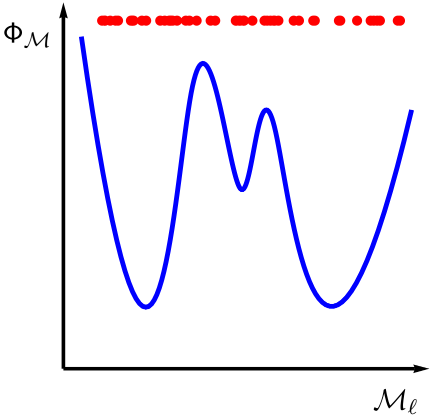
Unique best solution



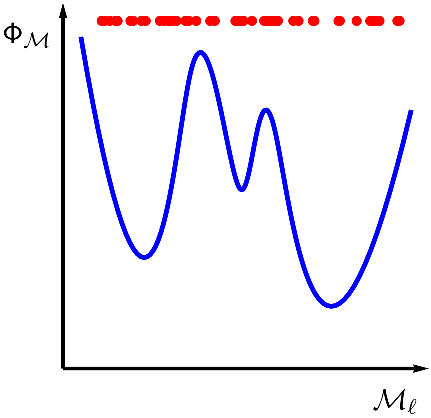
The $\Phi_{\mathcal{M}}$ is defined by the fitted Legendre coefficients $\left(a_L^{\mathrm{Fit}}\right)_k^{\alpha}$.

Cut selections for solution “data”

Mathematical ambiguity



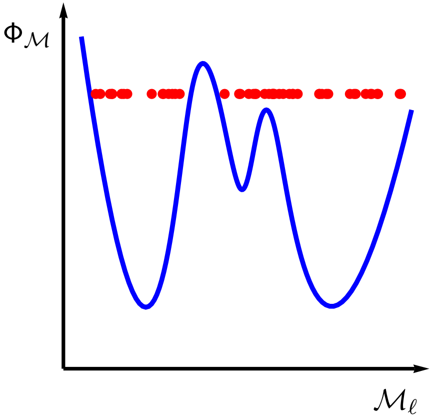
Unique best solution



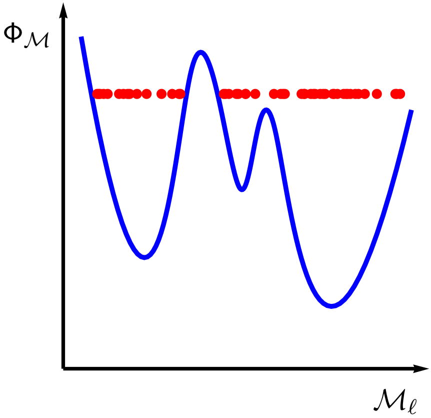
Start values have been distributed on the relevant part of the space $\mathcal{M}_\ell$.

Cut selections for solution "data"

Mathematical ambiguity

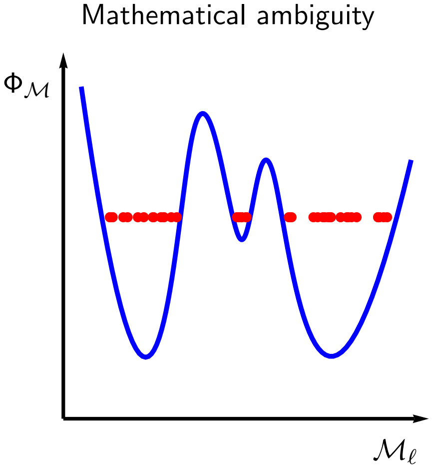


Unique best solution



Minimizations of $\Phi_{\mathcal{M}}$ converge within several iterations.

Cut selections for solution “data”



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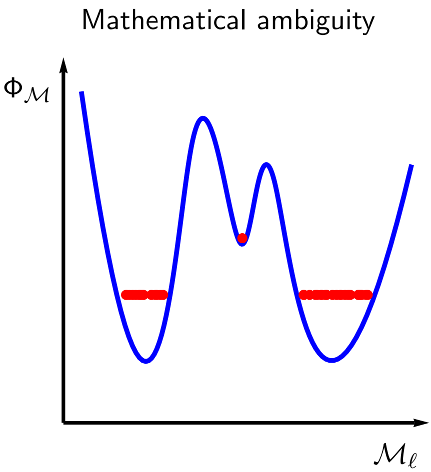

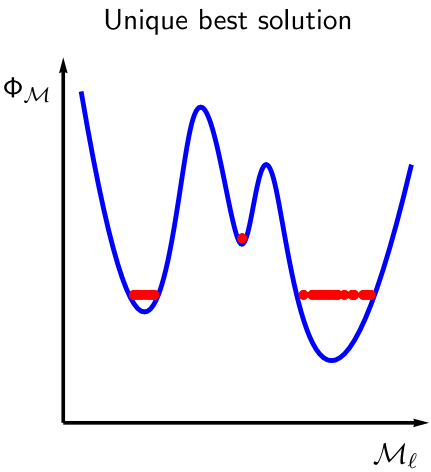

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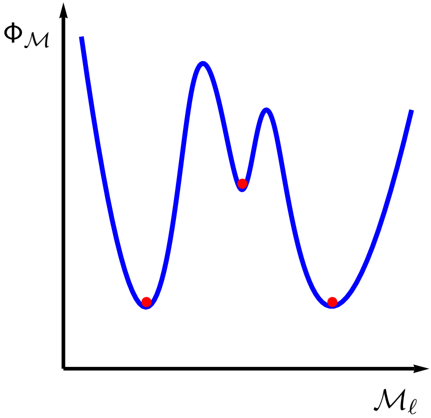

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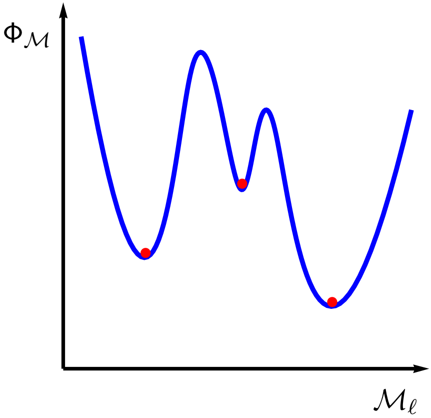

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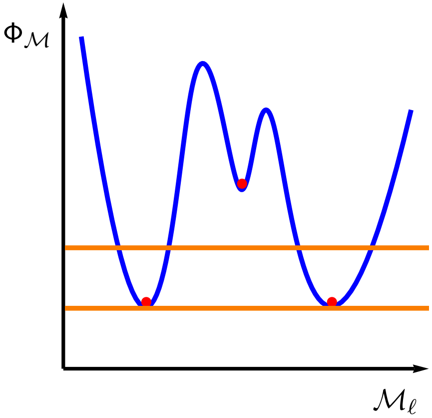
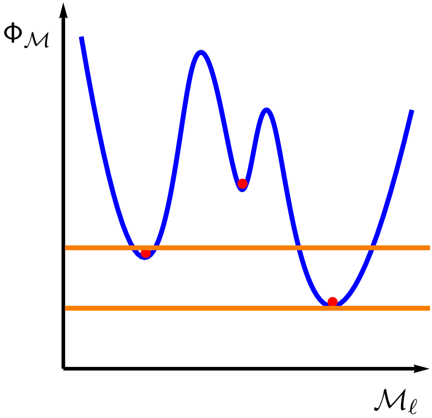
Mathematical ambiguity



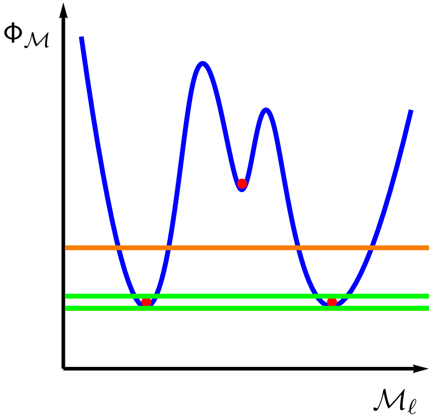
Unique best solution



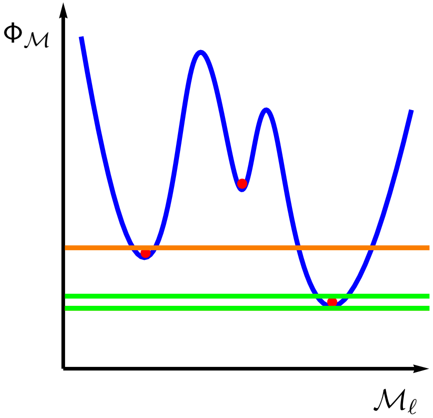
Cut selection using $\epsilon = 1$

Cut selections for solution “data”

Mathematical ambiguity



Unique best solution



Cut selection using $\epsilon = 1$ / Cut selection using $\epsilon \sim \mathrm{num.\,precision}$

Fits to truncated MAID theory data I

Fit of group S observables $\{\sigma_0, \Sigma, T, P\}$ generated using MAID2007 multipoles ($\gamma p \to \pi^0 p$) up to $\ell_{\max} = 1$ (Fit $\ell_{\max} = 1$, $N_{MC} = 1000$):

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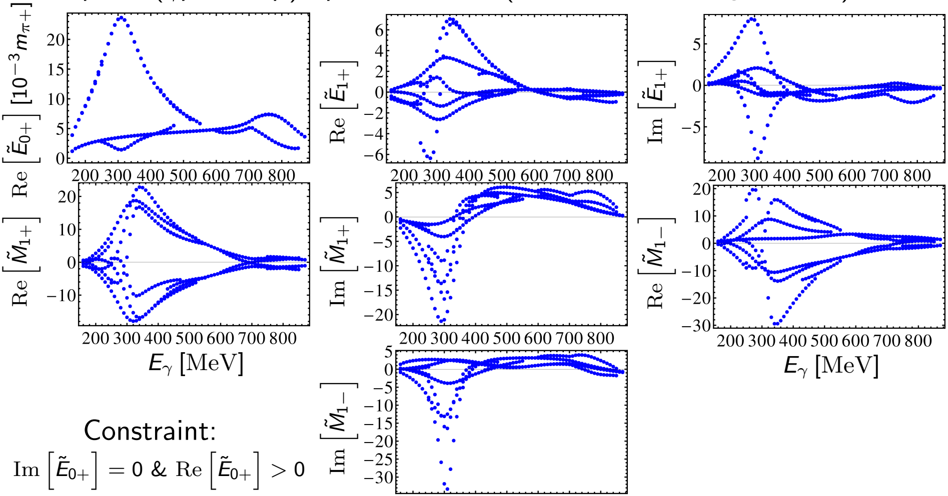

Constraint:

$\mathrm{Im}\left[\tilde{E}_{0+}\right] = 0$ & $\mathrm{Re}\left[\tilde{E}_{0+}\right] > 0$

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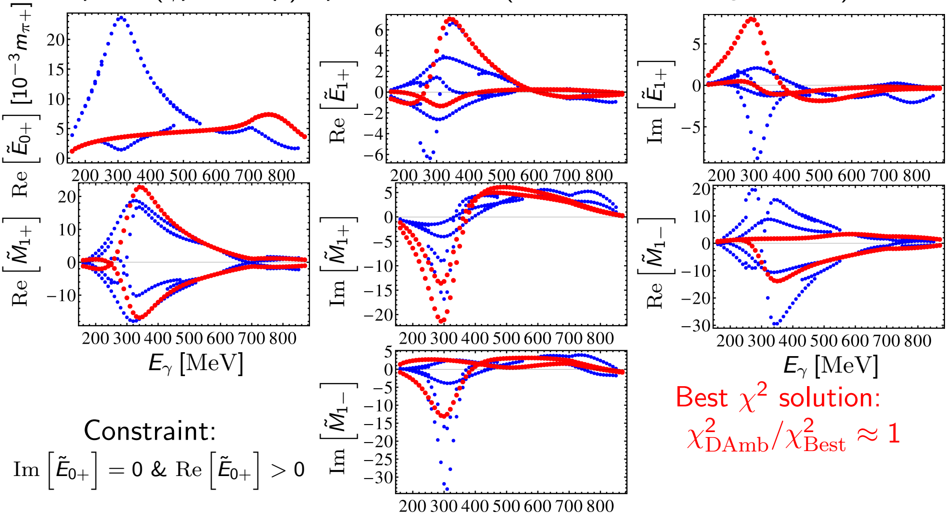

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Best χ^2 solution:

$\chi^2_{\mathrm{DAmb}} / \chi^2_{\mathrm{Best}} \approx 1$

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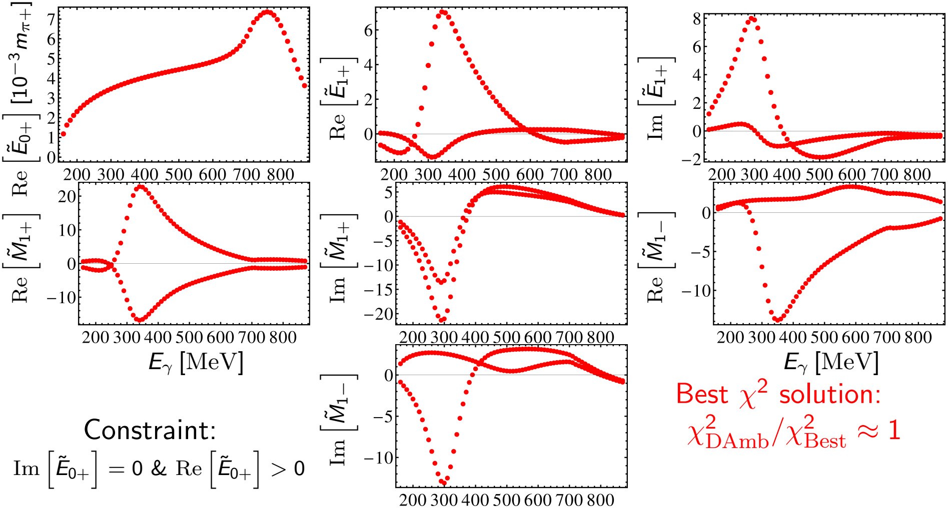

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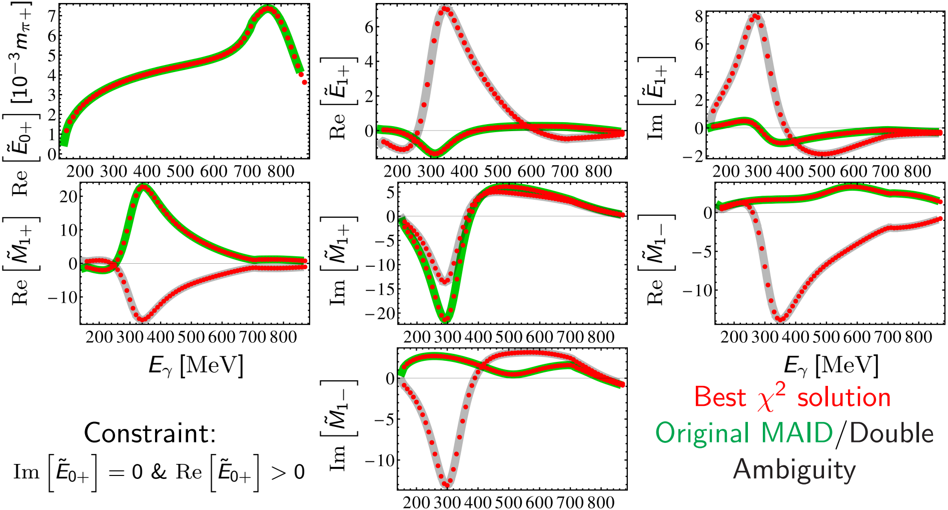


Constraint:

$\mathrm{Im}\left[\tilde{E}_{0+}\right] = 0$ & $\mathrm{Re}\left[\tilde{E}_{0+}\right] > 0$

Best χ^2 solution

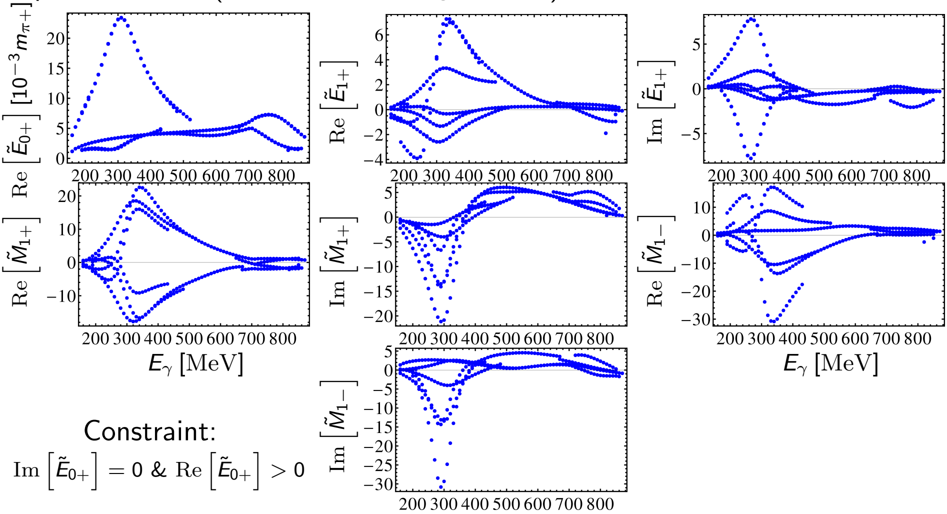
Original MAID/Double Ambiguity

Fits to truncated MAID theory data II

Fit of observables $\{\sigma_0, \Sigma, T, P, G\}$ generated using MAID2007 multipoles up to $\ell_{\max} = 1$ (Fit $\ell_{\max} = 1$, $N_{MC} = 1000$):

Fits to truncated MAID theory data II

Fit of observables $\{\sigma_0, \Sigma, T, P, G\}$ generated using MAID2007 multipoles up to $\ell_{\max} = 1$ (Fit $\ell_{\max} = 1$, $N_{MC} = 1000$):



Constraint:

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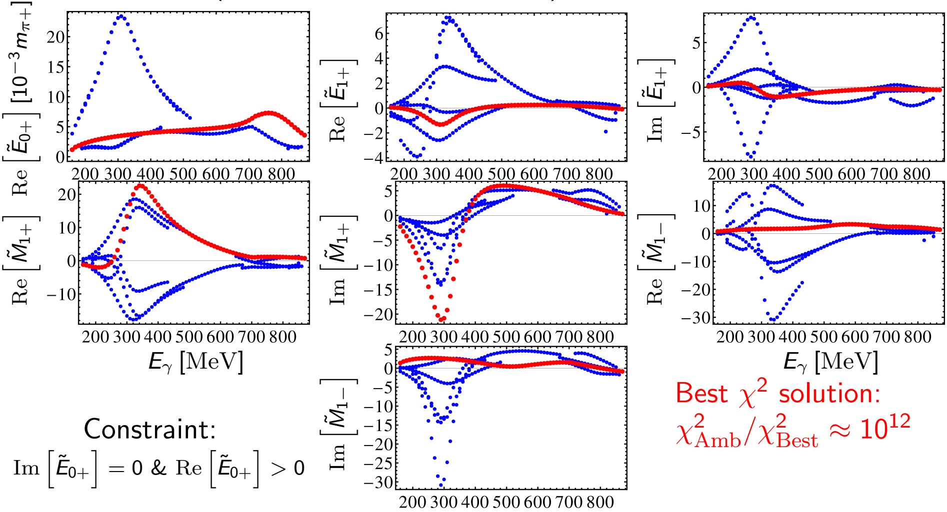
Fits to truncated MAID theory data II

Fit of observables $\{\sigma_0, \Sigma, T, P, G\}$ generated using MAID2007 multipoles up to $\ell_{\max} = 1$ (Fit $\ell_{\max} = 1$, $N_{MC} = 1000$):



Constraint:

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Best χ^2 solution:

$\chi^2_{\mathrm{Amb}}/\chi^2_{\mathrm{Best}} \approx 10^{12}$

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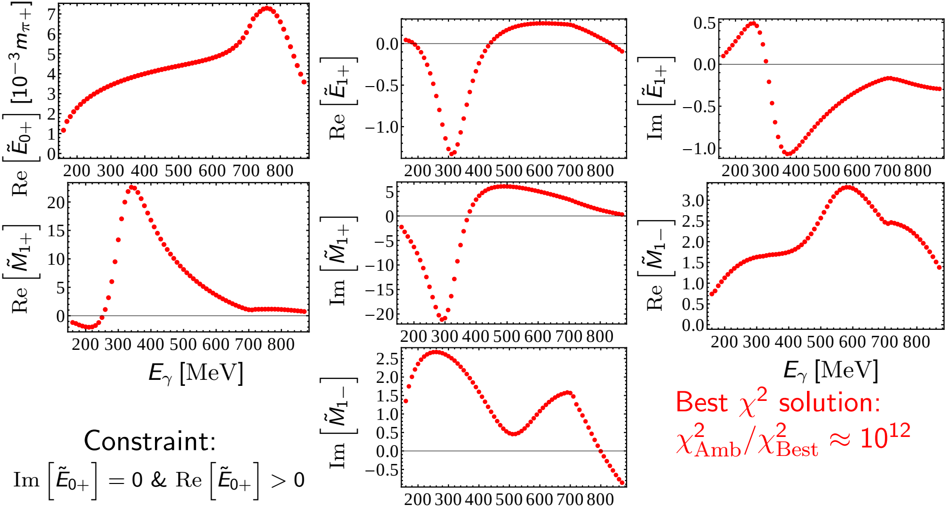

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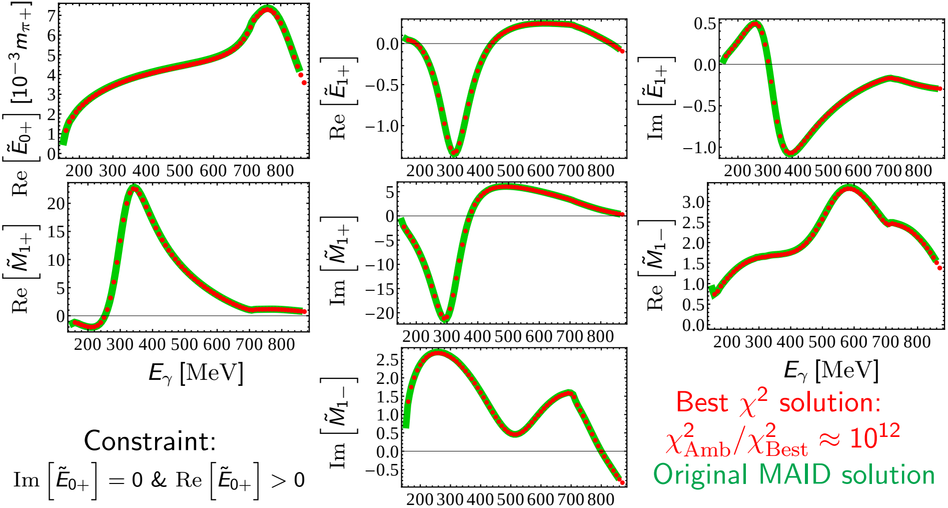

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Original MAID solution

Next step: fits to data

The following datasets were investigated for the process $\gamma p \rightarrow \pi^0 p$ in the Δ-region:

I. Data taken at the MAMI facility:

- σ_0: 114 energy points for $E_\gamma^{\mathrm{LAB}} \in [146.950, 420.270]$ MeV
  $\Delta\sigma_0 \leq 1\%$, [D. Hornidge et al., PRL 111 (2013) 062004]
- Σ: 67 energy points for $E_\gamma^{\mathrm{LAB}} \in [146.950, 440]$ MeV
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$\rightarrow$ 12073 data points available for $\gamma p \rightarrow \pi^0 p$.

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$\rightarrow$ 12073 data points available for $\gamma p \rightarrow \pi^0 p$.

$\rightarrow$ 3981 points used effectively in TPWA for $E_{\gamma}^{\mathrm{LAB}} = 280 \ldots 420$ MeV.

The method for fitting to real data

Question: Is there a method to investigate the influence of experimental errors on the results and uniqueness of TPWA fits?

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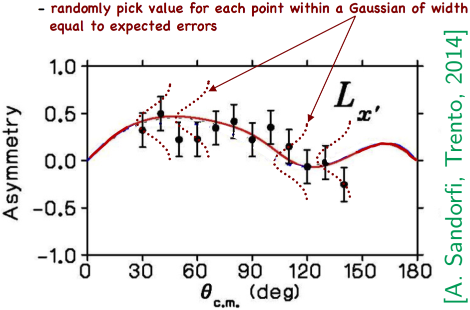
→Yes: Generate mock experiments using gaussian PDF, starting from original data or pseudodata ("bootstrapping")

⇒ Ensemble of datasets equivalent to the original data



[A. Sandorfi, Trento, 2014]

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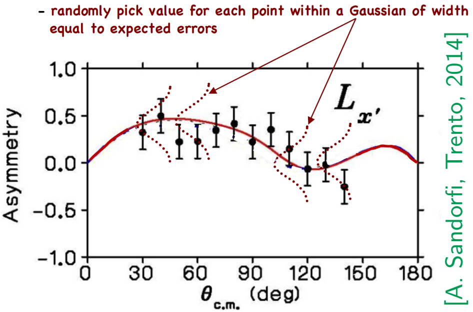


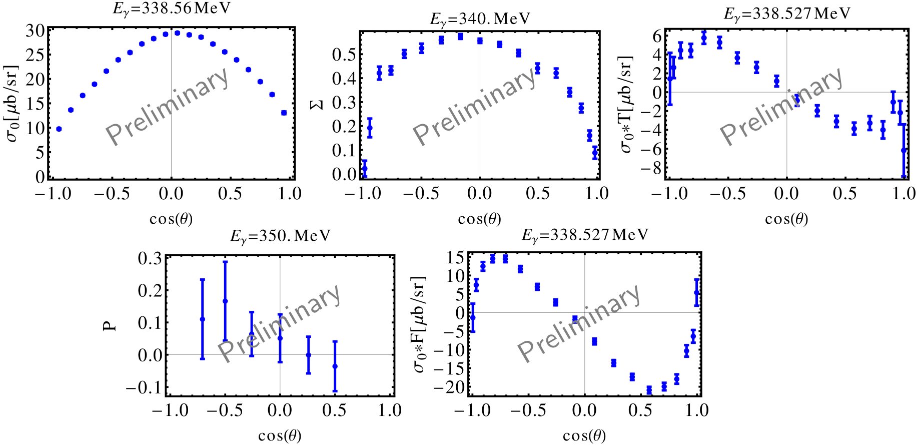
[A. Sandorfi, Trento, 2014]

→ Perform a model independent TPWA fit for each generated dataset
- Monte Carlo scan of the relevant amplitude space

→ Investigate presence of ambiguities. / Extract values and uncertainty bands for the multipoles in case no ambiguities are present.

TPWA fits using the bootstrapping method

$$E_\gamma^{\mathrm{LAB}} \simeq E_\gamma^{\Delta} \simeq 338 \text{ MeV}$$



Angular distributions of data as provided are shown.

TPWA fits using the bootstrapping method

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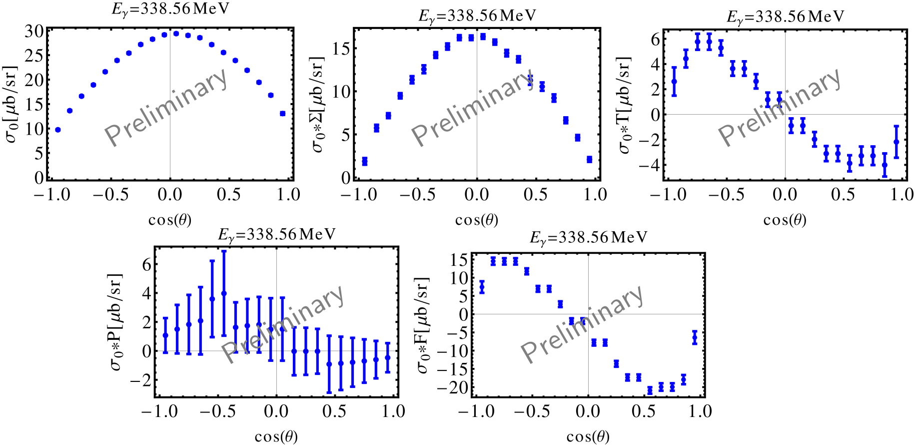

The data are re-binned to the kinematic grid dictated by the σ_0 measurement. Profile functions are calculated.

TPWA fits using the bootstrapping method

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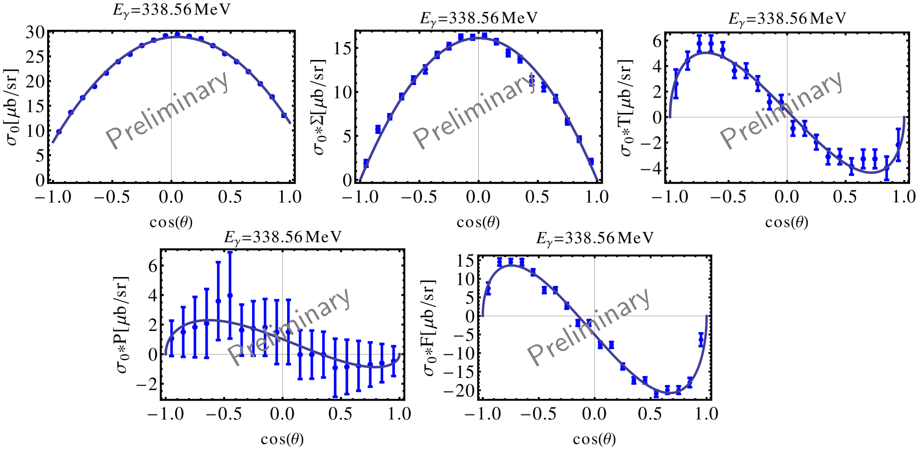

Profile functions for the original dataset are fitted with an S- and P-wave truncation ($\ell_{\max} = 1$).

TPWA fits using the bootstrapping method

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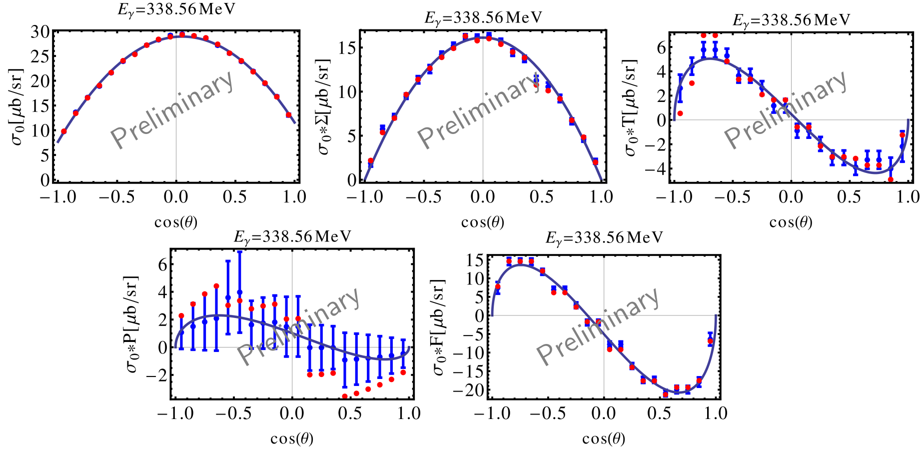
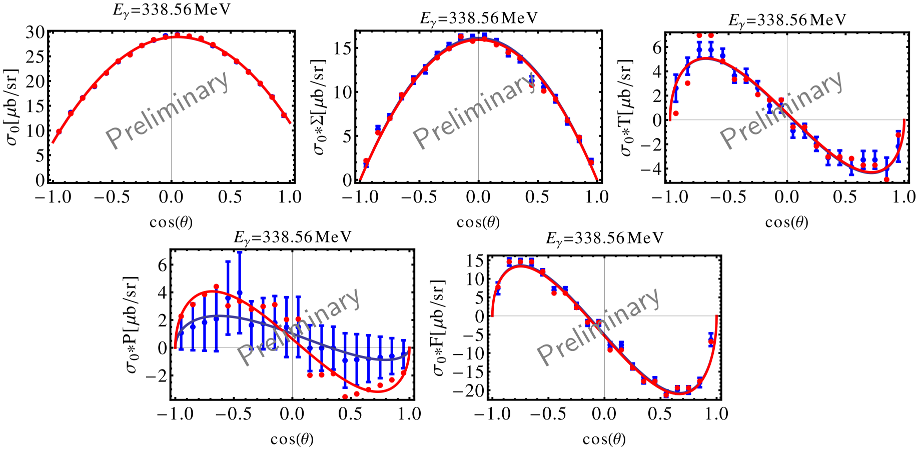


Generate 1 additional dataset using gaussian PDFs.

TPWA fits using the bootstrapping method

$$E_\gamma^{\mathrm{LAB}} \simeq E_\gamma^{\Delta} \simeq 338 \text{ MeV}$$



Fit the additional dataset.

TPWA fits using the bootstrapping method

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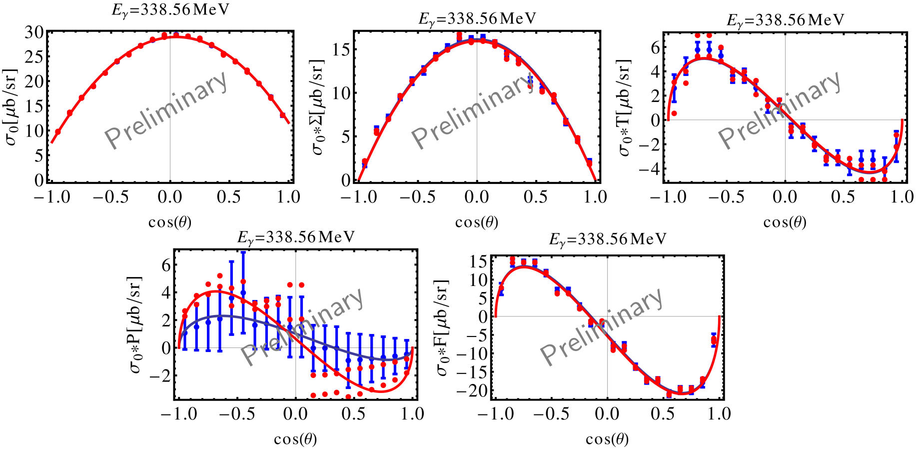

Generate 1 more dataset.

TPWA fits using the bootstrapping method

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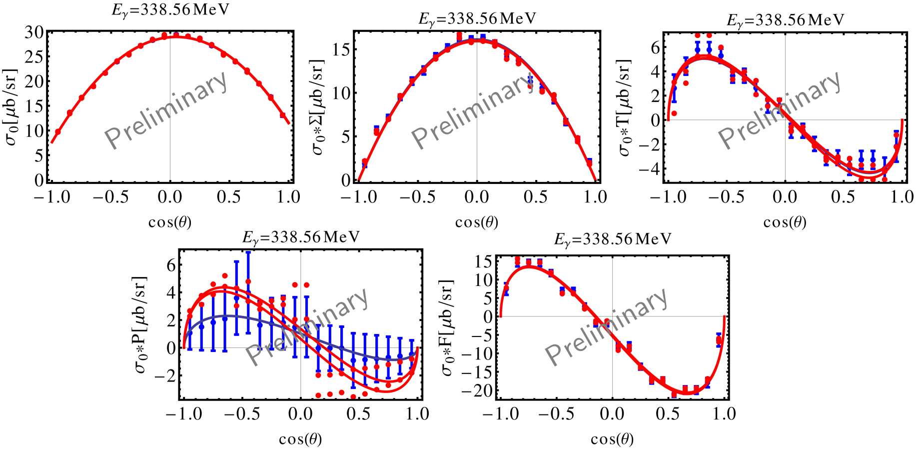

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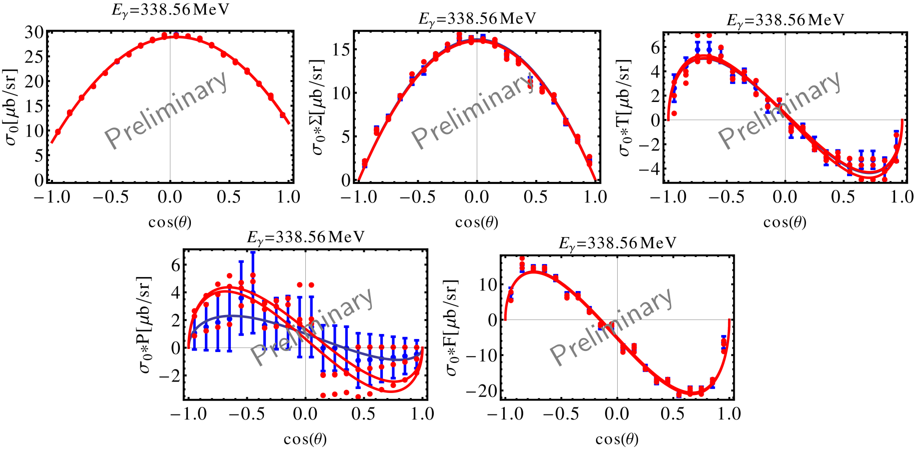

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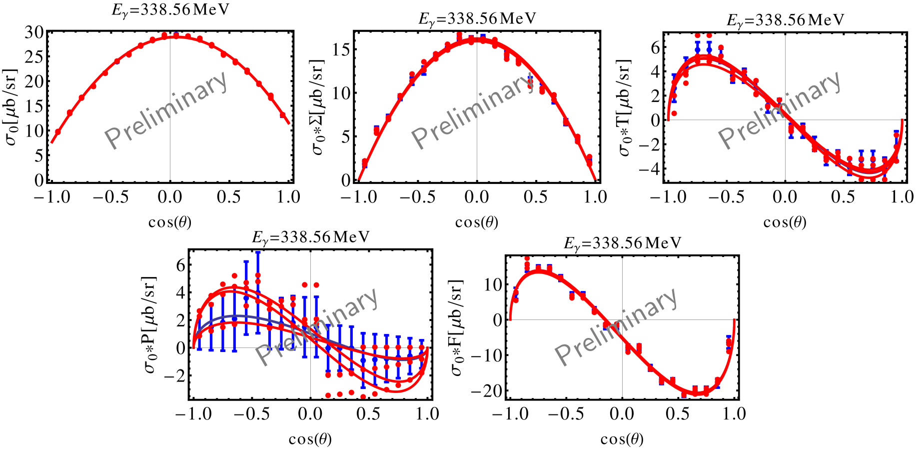

Fit the additional dataset.

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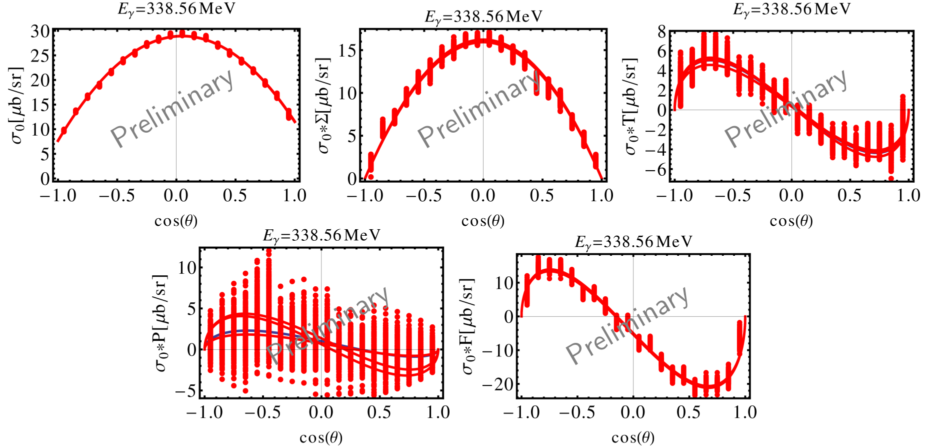
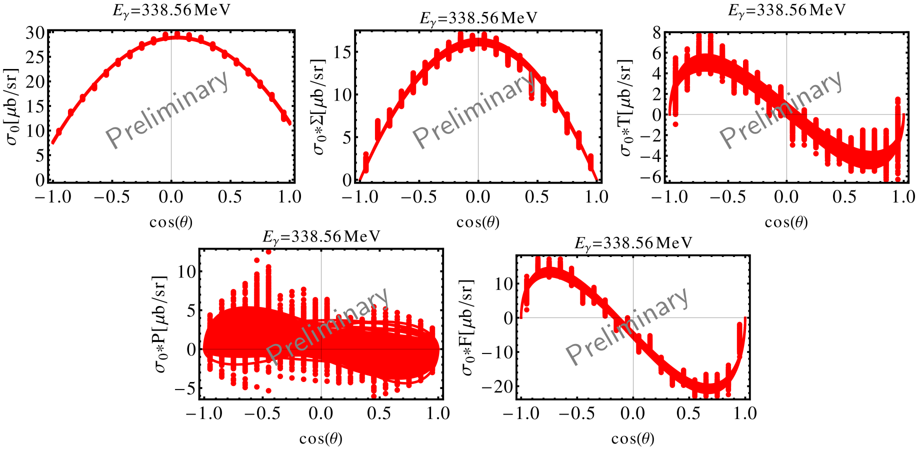


In total, 250 additional datasets are generated.

TPWA fits using the bootstrapping method

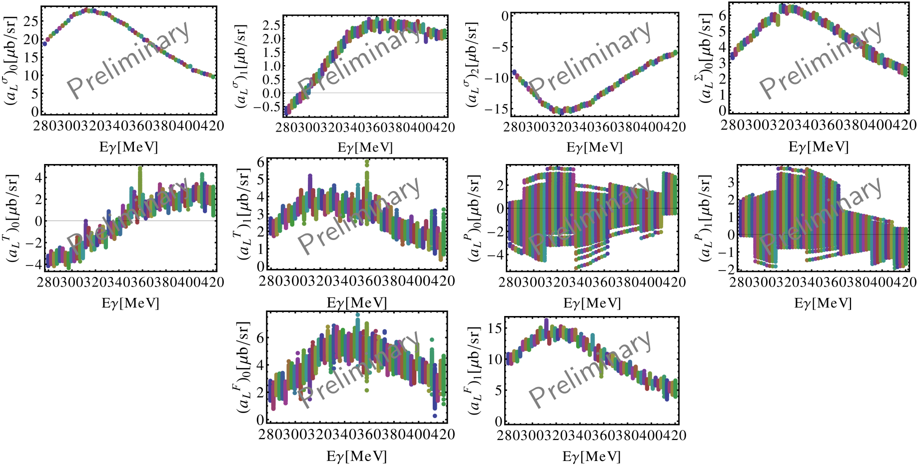
$$E_\gamma^{\mathrm{LAB}} \simeq E_\gamma^{\Delta} \simeq 338 \text{ MeV}$$



All of the $(1 + 250)$ datasets are fitted.

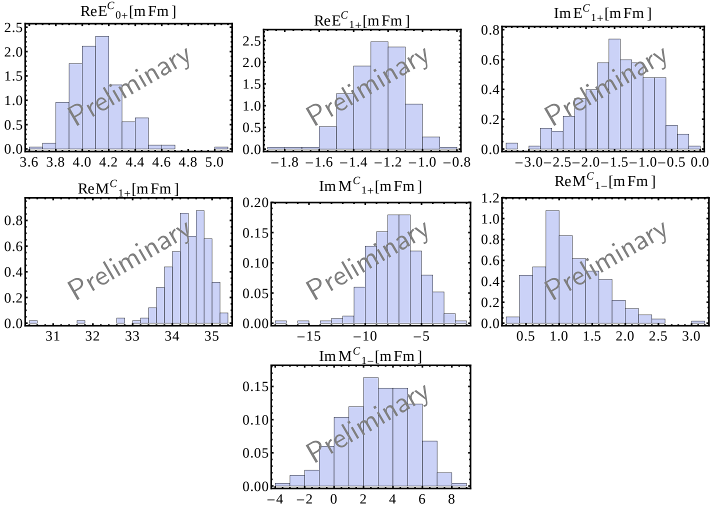
The TPWA fit step 2 is then applied to each one (for $\ell_{\max} = 1$).

Results for bootstrapped Legendre coefficients



Results for fits to real data I

$\gamma p \to \pi^0 p$: $\underline{\{\sigma_0, \Sigma, T, F\}}$ from MAMI and $\underline{P}$ from World Data.
Histogram results for an Ensemble of $(1 + 250)$ datasets at $E_{\gamma}^{\mathrm{LAB}} \simeq 338\,\mathrm{MeV}$:



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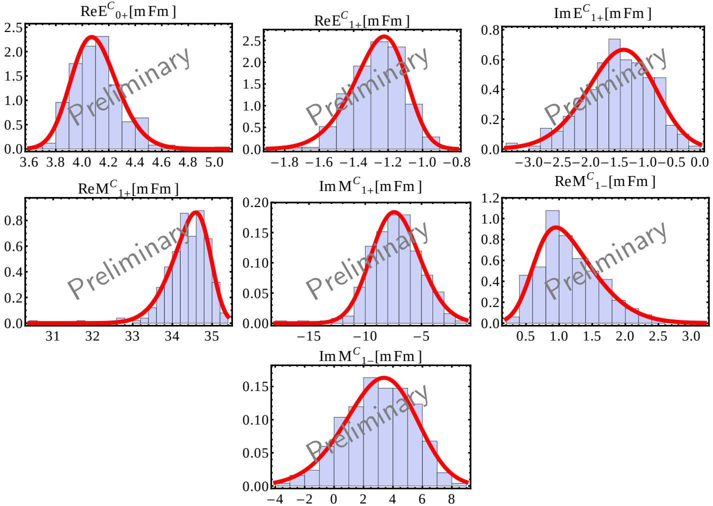

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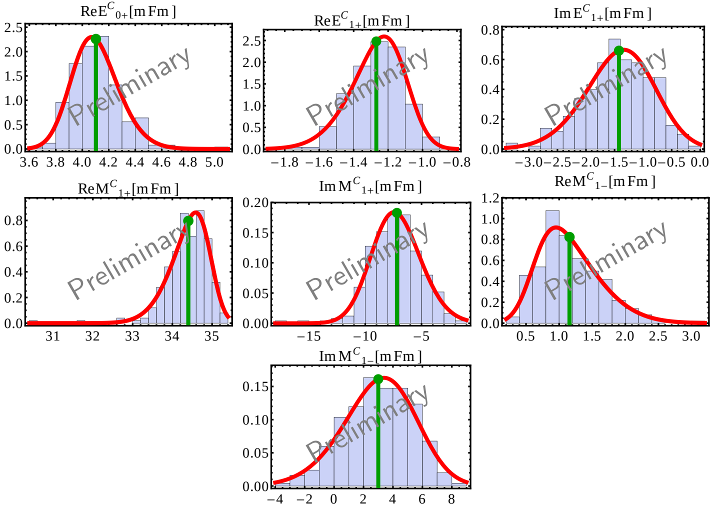

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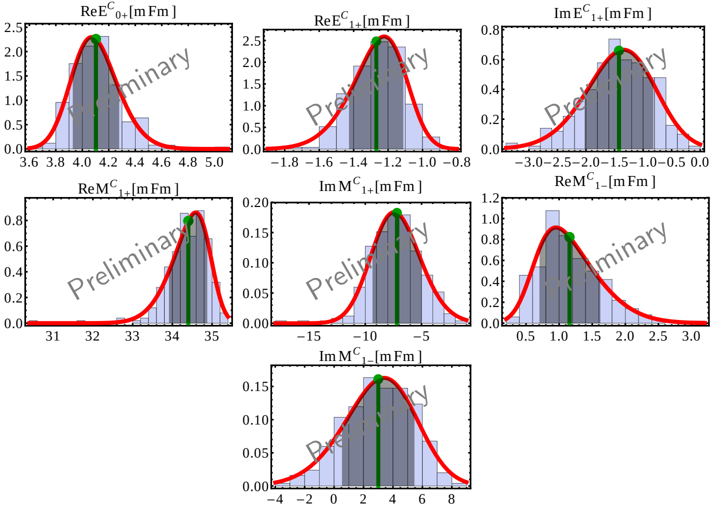
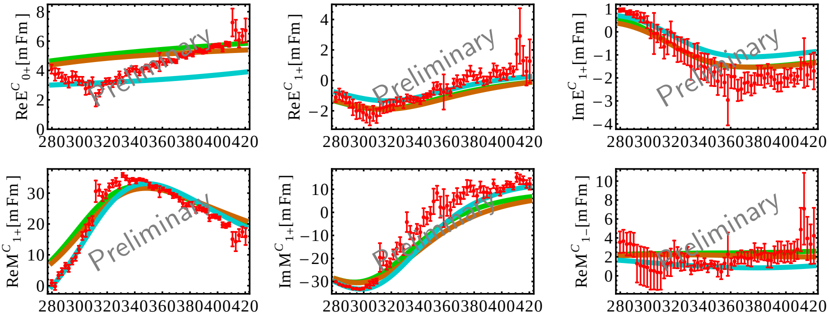


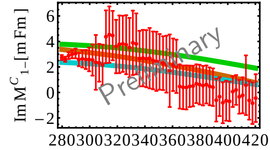
Results for fits to real data II

It is possible to verify the completeness of $\underline{\{\sigma_0, \Sigma, T, P, F\}}$ by fitting new MAMI data as well as $\underline{P}$-data from the world database for $\gamma p \to \pi^0 p$:



(1+250) Ensemble data sets

$N_{MC} = 1000$



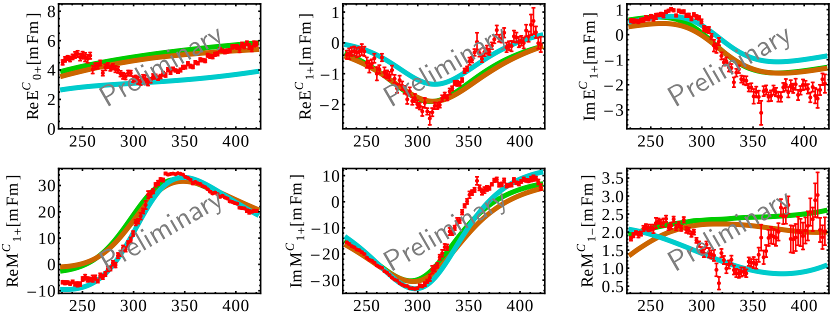
Green: MAID

Brown: SAID

Cyan: BnGa

Fits to MAMI- plus SAID-model data

Exchange old P-data for a SAID-prediction with 5%-errors. Use bootstrap with the Monte Carlo method as described before.



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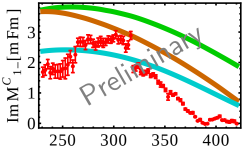

Green: MAID

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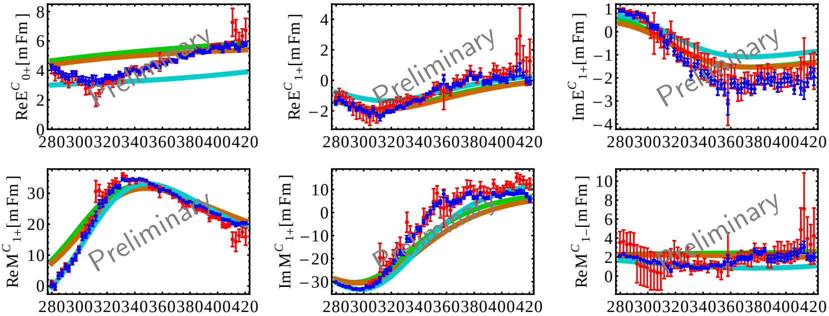
Comparison I

Compare results for the (MAMI+Belyaev)-dataset with those for the (MAMI+P^{SAID})-dataset.



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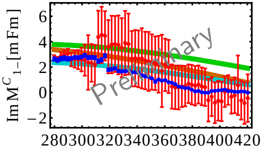


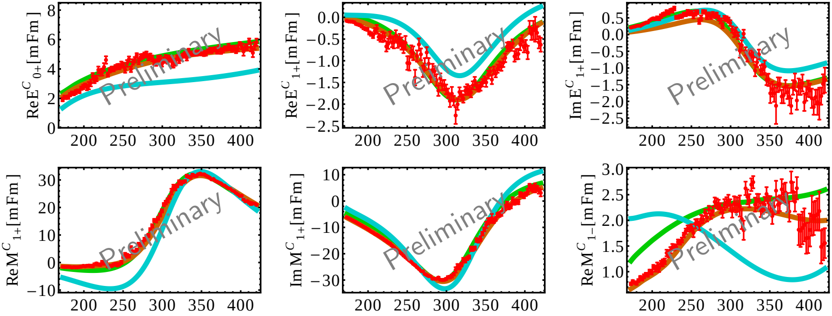
Green: MAID

Brown: SAID

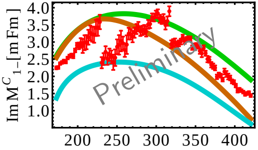
Cyan: BnGa

SAID-D-Waves included into fit

Fit (MAMI+P^{SAID})-data and include the D-waves from SAID as fixed parameters. Start from the <u>SAID</u> S- and P-wave multipoles in each fit.



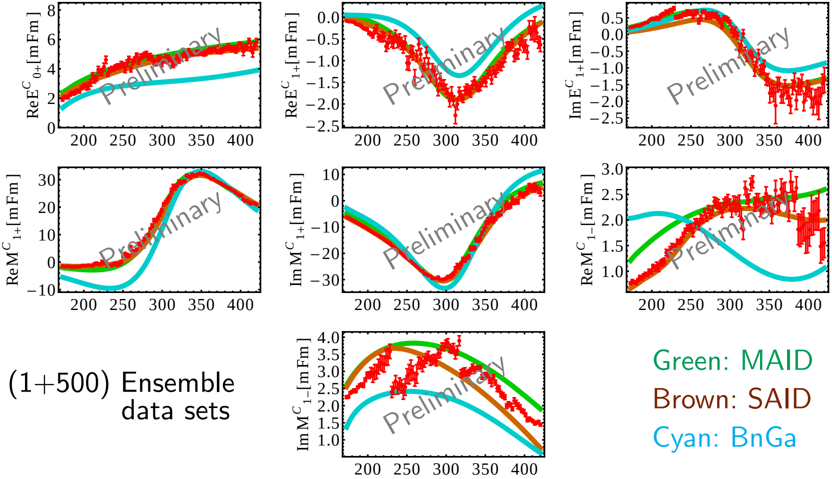
(1+500) Ensemble data sets



Green: MAID
Brown: SAID
Cyan: BnGa

SAID-D-Waves included into fit

Fit (MAMI+P^{SAID})-data and include the D-waves from SAID as fixed parameters. Start from the BnGa S- and P-wave multipoles in each fit.

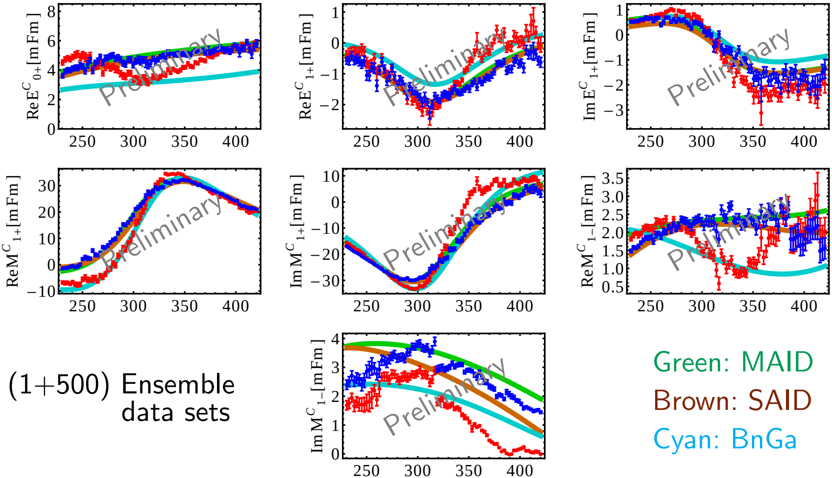


(1+500) Ensemble data sets

Green: MAID
Brown: SAID
Cyan: BnGa

Comparison II

The (MAMI+P^{SAID})-dataset is fitted. Compare fits without and including D-waves from SAID.

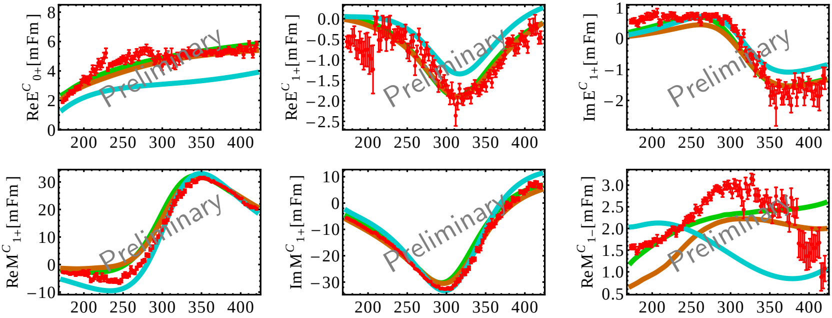


(1+500) Ensemble data sets

Green: MAID
Brown: SAID
Cyan: BnGa

BnGa-D-Waves included into fit

Fit (MAMI+P^{SAID})-data and include the D-waves from BnGa as fixed parameters. Start from the <u>BnGa</u> S- and P-wave multipoles in each fit.



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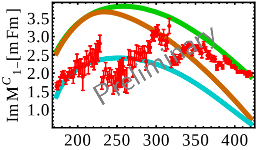

Green: MAID
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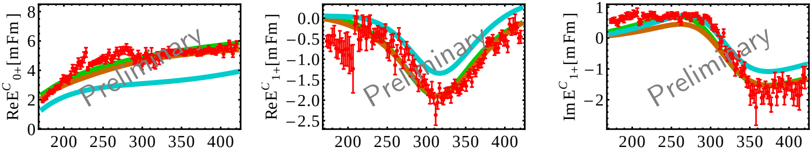
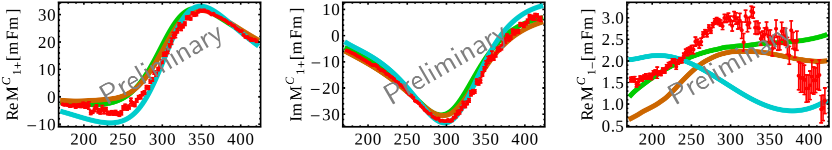
Cyan: BnGa

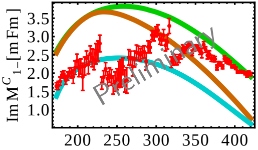
BnGa-D-Waves included into fit

Fit (MAMI+P^{SAID})-data and include the D-waves from BnGa as fixed parameters. Start from the <u>SAID</u> S- and P-wave multipoles in each fit.





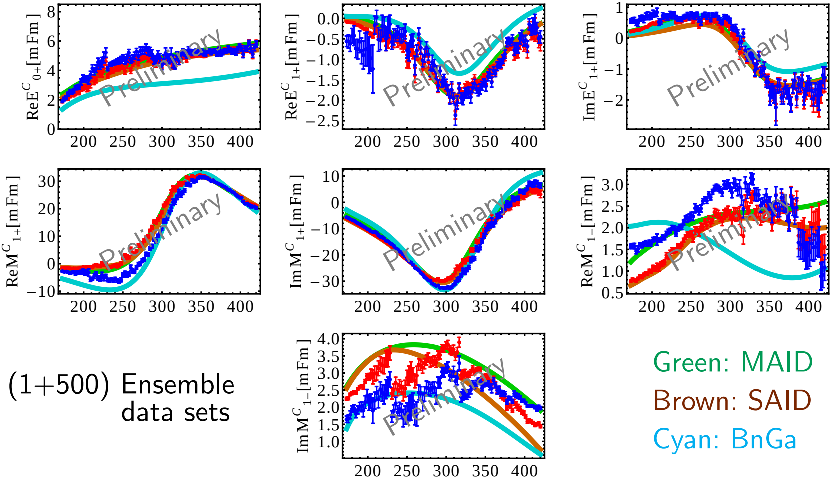
(1+500) Ensemble data sets



Green: MAID
Brown: SAID
Cyan: BnGa

Comparison III

For the (MAMI+P^{SAID})-dataset, compare fits including SAID-D-waves and BnGa-D-waves as fixed parameters.



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Conclusions & Outlook

I. The result of Chiang/Tabakin has been verified: 8 observables can yield a Complete Experiment.

II. The unknown phase $\phi^{F}\left(W,\theta\right)$ denies access to partial waves.

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$\rightarrow$ Only 5 observables (theoretically) necessary, examples:

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IV. Numerical TPWA fits of mock data (from predictions, e.g. MAID) as well as real data (MAMI) confirm certain complete experiments. The bootstrapping method ($\simeq$ surrogate data testing) has been proposed to check for ambiguities and extract multipoles with errors.

$\rightarrow$ Investigate further datasets over broader energy regions as well as for higher $\ell_{\max}$. Check for ambiguities, dependence on inclusion of model multipoles, ...

Points for discussion / Questions

→ Advantages/disadvantages of the bootstrapping method (possible pitfalls)?

→ Fit with/without correlations? I.e. defining $i = (\alpha, k)$, $j = (\alpha', k')$,

$$\chi^2 (\mathcal{M}_\ell) = \sum_{i,j} \Big[\left(a_L^{\mathrm{Fit}}\right)_i - \langle \mathcal{M}_\ell | \left(C_L\right)_i | \mathcal{M}_\ell \rangle \Big] \mathrm{C}_{ij}^{-1} \Big[\left(a_L^{\mathrm{Fit}}\right)_j - \langle \mathcal{M}_\ell | \left(C_L\right)_j | \mathcal{M}_\ell \rangle \Big],$$

with covariance matrix C stemming from simultaneous angular fit of all used observables.

→ Calculate/estimate covariance and correlation matrix for multipole results from bootstrap, just using

$$\mathrm{C}\left(X, Y\right) = \left\langle \left(X - \langle X \rangle\right) \left(Y - \langle Y \rangle\right) \right\rangle, \; C^R \left(X, Y\right) = \frac{\mathrm{C}(X,Y)}{\Delta X \, \Delta Y}.$$

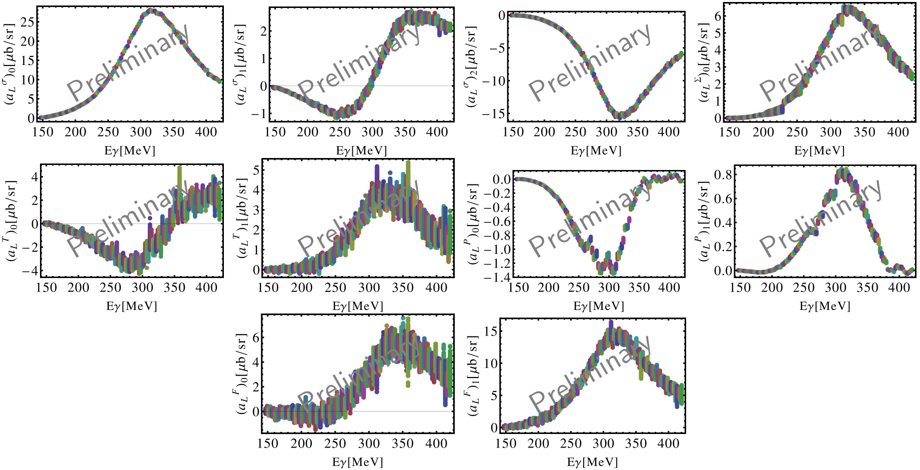
→ What is the reason for the apparent reduction of the complete experiments

$$8 \left\{ \text{extraction of the } F_i \left(W, \theta\right) \right\} \longrightarrow 5 \left\{ \text{TPWA} \right\} ?$$

((fixed-s) analyticity, truncation, spin physics (i.e. definition of transversity amplitudes $b_i \left(W, \theta\right)$), …?)

Thank You!

Appendices: Results for Legendre coefficients - (MAMI+P^{SAID})



Appendices: Does not knowing $\phi^F(W, \theta)$ cause harm?

I. For non. rel. QM / Spinless scattering:

$$f(W,\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(W) P_\ell(\cos\theta) \leftrightarrow f_\ell(W) = \frac{1}{2}\int_{-1}^{1} d\cos\theta f(W,\theta) P_\ell(\cos\theta)$$

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II. There exist more involved projections for photoproduction, e.g.:

$$M_{\ell+}(W) = \frac{1}{2(\ell+1)} \int_{-1}^{1} d\cos\theta \Big[F_1(W, \theta) P_\ell(\cos\theta) - F_2(W, \theta) P_{\ell+1}(\cos\theta) - F_3(W, \theta) \frac{P_{\ell-1}(\cos\theta) - P_{\ell+1}(\cos\theta)}{2\ell + 1} \Big]$$

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$\rightarrow$ Not knowing $\phi^F(W,\theta)$ denies access to partial waves via the full amplitudes!

Appendices: Counting real degrees of freedom I

- The maximal $\cos\theta$ powers in the CGLN amplitudes are:

  $F_1 \sim (\cos\theta)^{\ell_{\mathrm{max}}}$, $F_2 \sim (\cos\theta)^{\ell_{\mathrm{max}}-1}$, $F_3 \sim (\cos\theta)^{\ell_{\mathrm{max}}-1}$, $F_4 \sim (\cos\theta)^{\ell_{\mathrm{max}}-2}$.

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$\rightarrow$ Example: differential cross section σ_0

$$\sigma_0 = \mathrm{Re}\Big[\left|F_1\right|^2 + \left|F_2\right|^2 - 2\cos(\theta) F_1^{*} F_2 + \frac{1}{2} \sin^2(\theta) \Big\{ \left|F_3\right|^2 + \left|F_4\right|^2 + 2 F_1^{*} F_4 + 2 F_2^{*} F_3 + 2\cos(\theta) F_3^{*} F_4 \Big\} \Big].$$

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$$+ \underbrace{|F_4|^2}_{\sim\cos\theta^{2\ell_{\mathrm{max}}-4}} + 2\underbrace{F_1^* F_4}_{\sim\cos\theta^{2\ell_{\mathrm{max}}-2}} + 2\underbrace{F_2^* F_3}_{\sim\cos\theta^{2\ell_{\mathrm{max}}-2}} + 2\underbrace{\cos(\theta) F_3^* F_4}_{\sim\cos\theta^{2\ell_{\mathrm{max}}-2}} \Big\}\Big].$$

Therefore: $\sigma_0 \sim (\cos\theta)^{2\ell_{\mathrm{max}}}$

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Therefore: $\sigma_0 \sim (\cos\theta)^{2\ell_{\max}}$

Count maximal $\cos\theta$ powers for group S and $\mathcal{BT}$ observables:

$\sigma_0 \sim (\cos\theta)^{2\ell_{\max}}$ $\check{\Sigma} \sim (\cos\theta)^{2\ell_{\max}-2}$ $\check{T} \sim (\cos\theta)^{2\ell_{\max}-1}$ $\check{P} \sim (\cos\theta)^{2\ell_{\max}-1}$

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Appendices: Counting real degrees of freedom II

Add $+1$ for $(\cos\theta)^0$-term in order to obtain:

Number of angular fit coefficients a_k^α provided by group S and $\mathcal{BT}$:

$\sigma_0 \sim (2\ell_{\mathrm{max}} + 1)$ $\check{\Sigma} \sim (2\ell_{\mathrm{max}} - 1)$ $\check{T} \sim 2\ell_{\mathrm{max}}$ $\check{P} \sim 2\ell_{\mathrm{max}}$
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I. The number of real parameters to be determined in a TPWA is:

$$\underbrace{4\ell_{\mathrm{max}}}_{\#\,\mathrm{of}\,\mathcal{M}_\ell} \times \underbrace{2}_{\mathcal{M}_\ell \in \mathbb{C}} - \underbrace{1}_{\text{overall phase fixed}} = (8\ell_{\mathrm{max}}-1)$$

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II. Compare number of a_k^α to the number of varied parameters:

- $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\}$: $8\ell_{\mathrm{max}}$ $[a_k^\alpha] > (8\ell_{\mathrm{max}} - 1)$, however: discrete ambiguities!

Appendices: Counting real degrees of freedom II

Add $+1$ for $(\cos\theta)^0$-term in order to obtain:

Number of angular fit coefficients a_k^α provided by group S and $\mathcal{BT}$:

$\sigma_0 \sim (2\ell_{\mathrm{max}}+1)$ $\check{\Sigma} \sim (2\ell_{\mathrm{max}}-1)$ $\check{T} \sim 2\ell_{\mathrm{max}}$ $\check{P} \sim 2\ell_{\mathrm{max}}$
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- $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\} \oplus \check{E}$: $(10\ell_{\mathrm{max}}+1)$ $[a_k^\alpha] > (8\ell_{\mathrm{max}}-1)$, still discr. ambig.!

Appendices: Counting real degrees of freedom II

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- $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\} \oplus \check{F}$: $(10\ell_{\mathrm{max}})$ $[a_k^\alpha] > (8\ell_{\mathrm{max}} - 1)$, complete set.

Appendices: Counting real degrees of freedom II

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$\rightarrow$ Comparison of real degrees of freedom seems promising!

Appendices: TPWA for Photoproduction I

For the reaction $\gamma N \rightarrow \varphi B$, there are 16 (polarization) observables. Written using CGLN amplitudes $\{F_i\,(W,\,\theta)\,, i = 1, \ldots, 4\}$, they take the form:

$$\check{\Omega}^{\alpha}\,(W, \theta) = \frac{1}{2}\,\langle F|\,\hat{A}^{\alpha}\,|F\rangle\,,\;\alpha = 1, \ldots, 16$$

$$\text{e.g.}:\;\check{\Sigma} = -\frac{\sin^2(\theta)}{2}\mathrm{Re}\left[|F_3|^2 + |F_4|^2 + 2\left\{F_1^* F_4 + F_2^* F_3 + \cos(\theta) F_3^* F_4\right\}\right]$$

Appendices: TPWA for Photoproduction I

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The observables simplify significantly once transversity amplitudes $\{b_i\left(W,\, \theta\right), i = 1, \ldots, 4\}$ are used:

$$\check{\Omega}^{\alpha}\left(W, \theta\right) = \frac{1}{2} \left\langle b \right| \tilde{\Gamma}^{\alpha} \left| b \right\rangle, \; \alpha = 1, \ldots, 16$$

$$\tilde{\Gamma}^{\alpha}: \; 16 \; \text{hermitean} \; 4 \times 4 \; \text{Gamma} - \text{matrices}$$

$$\text{e.g.}: \; \check{\Sigma} = -\frac{1}{2}\left(\left|b_1\right|^2 + \left|b_2\right|^2 - \left|b_3\right|^2 - \left|b_4\right|^2\right)$$

[Chiang/Tabakin(1996)]

Appendices: TPWA for Photoproduction II

- Problem: What minimum subsets of observables are necessary in order to extract the multipoles $\{E_{\ell\pm}(W),\ M_{\ell\pm}(W)\}$ appearing in the truncated partial wave expansion of for example CGLN amplitudes:

$$F_1(W,\theta) = \sum_{\ell=0}^{\ell_{\mathrm{max}}} \Big\{ [\ell M_{\ell+} + E_{\ell+}] P^{'}_{\ell+1}(\cos(\theta)) + [(\ell+1) M_{\ell-} + E_{\ell-}] P^{'}_{\ell-1}(\cos(\theta)) \Big\},$$

$$F_2(W,\theta) = \sum_{\ell=1}^{\ell_{\mathrm{max}}} [(\ell+1) M_{\ell+} + \ell M_{\ell-}] P^{'}_{\ell}(\cos(\theta)),$$

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$$F_4(W,\theta) = \sum_{\ell=2}^{\ell_{\mathrm{max}}} [M_{\ell+} - E_{\ell+} - M_{\ell-} - E_{\ell-}] P^{''}_{\ell}(\cos(\theta)),$$

for some finite ℓ_{max}?

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  for some finite $\ell_{\max}$?

- Extraction only unambiguous up to one energy dependent overall phase $\Phi(W)$ for all multipoles.

Appendices: TPWA for Photoproduction III

- Truncation at some finite value $\ell = \ell_{\mathrm{max}}$ leads to an angular parametrization of observables:

$$\check{\Omega}^{\alpha}\left(W,\theta\right) = \sum_{k=\beta_{\alpha}}^{2\ell_{\mathrm{max}}+\beta_{\alpha}+\gamma_{\alpha}} \left(a_{L}\right)_{k}^{\alpha}\left(W\right) P_{k}^{\beta_{\alpha}}\left(\cos\theta\right),$$

$$\left(a_{L}\right)_{k}^{\alpha}\left(W\right) = \sum_{\ell,\ell'=0}^{\ell_{\mathrm{max}}} \sum_{\kappa,\kappa'=1}^{4} \mathcal{C}_{\ell,\ell'}^{\kappa,\kappa'} \mathcal{M}_{\ell,\kappa}^{*}\left(W\right) \mathcal{M}_{\ell',\kappa'}\left(W\right),$$

involving associated Legendre polynomials $P_{l}^{m}\left(\cos\theta\right)$.

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Appendices: TPWA for Photoproduction III

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- Truncated partial wave analysis (TPWA):
  1. Legendre polynomial fit to angular distributions yields $\left(a_{L} \right)_{k}^{\alpha} \left(W \right)$
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  2. $\left(a_L \right)_{k}^{\alpha} \left(W \right)$ are solved for multipoles (up to an overall phase)
- Problem similar to full amplitude complete experiment, though dimension of matrices representing $\left(a_L \right)_{k}^{\alpha} \left(W \right)$ increases with ℓ_{max}.

Appendices: Derivation of product representations I

- [Omelaenko(1981)]: Formalism treating ambiguities in a TPWA.

Appendices: Derivation of product representations I

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- Starts considering the unpolarized CS and single spin observables (group S). These are most easily measured and have a simple form in the transversity representation:

Observable	Transversity representation	Type
$I(\theta) = \sigma_0/\rho$	$\frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
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- Derive a form of the $b_i(W, \theta)$ that is tailored to study ambiguities of the group S observables

  $\rightarrow$ Product representations

Appendices: Derivation of product representations II

- Use: $b_1(W,\theta) = b_2(W,-\theta)$ and $b_3(W,\theta) = b_4(W,-\theta)$.

Appendices: Derivation of product representations II

- Use: $b_1 (W, \theta) = b_2 (W, -\theta)$ and $b_3 (W, \theta) = b_4 (W, -\theta)$.
- Idea: exchange the angular variable $\cos\theta$ for $t = \tan\frac{\theta}{2}$ via

$$\cos\theta = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} \;\leftrightarrow\; \tan\frac{\theta}{2} = \begin{cases} +\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}, & \theta \in [0, \pi] \\ -\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}, & \theta \in [-\pi, 0] \end{cases}$$

Appendices: Derivation of product representations II

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- Legendre polynomials (and derivatives thereof) are hypergeometric functions of $-t^2$:

$$P_\ell (\cos\theta) = (1 + t^2)^{-\ell} \, {}_2F_1 \left(-\ell, -\ell; 1; -t^2\right), \ldots$$

Appendices: Derivation of product representations II

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$\rightarrow$ The transversity amplitudes become:

$$b_4 (W, \theta) = \mathcal{C} \, \frac{\exp\left(i\frac{\theta}{2}\right)}{(1 + t^2)^{\ell_{\mathrm{max}}}} \, A'_{2\ell_{\mathrm{max}}} (t) \,,$$

$$b_2 (W, \theta) = -\mathcal{C} \, \frac{\exp\left(i\frac{\theta}{2}\right)}{(1 + t^2)^{\ell_{\mathrm{max}}}} \, \left[A'_{2\ell_{\mathrm{max}}} (t) + t D'_{2\ell_{\mathrm{max}}-2} (t)\right] .$$

Appendices: Derivation of product representations III

- $A'_{2\ell_{\max}}(t) = \sum_{\ell=0}^{2\ell_{\max}} a_\ell t^\ell$ and
  $B'_{2\ell_{\max}}(t) = A'_{2\ell_{\max}}(t) + tD'_{2\ell_{\max}-2}(t) = \sum_{\ell=0}^{2\ell_{\max}} b_\ell t^\ell$ fulfill:
  $$a_{2\ell_{\max}} = b_{2\ell_{\max}} \quad \& \quad a_0 = b_0 .$$

Appendices: Derivation of product representations III

- $A'_{2\ell_{\mathrm{max}}}\left(t\right) = \sum_{\ell=0}^{2\ell_{\mathrm{max}}} a_{\ell} t^{\ell}$ and $B'_{2\ell_{\mathrm{max}}}\left(t\right) = A'_{2\ell_{\mathrm{max}}}\left(t\right) + t D'_{2\ell_{\mathrm{max}}-2}\left(t\right) = \sum_{\ell=0}^{2\ell_{\mathrm{max}}} b_{\ell} t^{\ell}$ fulfill:

$$a_{2\ell_{\mathrm{max}}} = b_{2\ell_{\mathrm{max}}} \quad \& \quad a_{0} = b_{0}.$$

- For the normalized polynomials $A_{2\ell_{\mathrm{max}}}\left(t\right) = \frac{A'_{2\ell_{\mathrm{max}}}\left(t\right)}{a_{2\ell_{\mathrm{max}}}}$ and $B_{2\ell_{\mathrm{max}}}\left(t\right) = \frac{B'_{2\ell_{\mathrm{max}}}\left(t\right)}{a_{2\ell_{\mathrm{max}}}}$ the equality of the free terms is valid (this will be important later on):

$$A_{2\ell_{\mathrm{max}}}\left(t = 0\right) = B_{2\ell_{\mathrm{max}}}\left(t = 0\right).$$

Appendices: Derivation of product representations III

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- For the normalized polynomials $A_{2\ell_{\max}}(t) = \frac{A'_{2\ell_{\max}}(t)}{a_{2\ell_{\max}}}$ and $B_{2\ell_{\max}}(t) = \frac{B'_{2\ell_{\max}}(t)}{a_{2\ell_{\max}}}$ the equality of the free terms is valid (this will be important later on):

$$A_{2\ell_{\max}}(t=0) = B_{2\ell_{\max}}(t=0) .$$

- $A_{2\ell_{\max}}(t)$ and $B_{2\ell_{\max}}(t)$ decompose into linear factors:

$$A_{2\ell_{\max}}(t) = \prod_{k=1}^{2\ell_{\max}} (t - \alpha_k) , \; B_{2\ell_{\max}}(t) = \prod_{k=1}^{2\ell_{\max}} (t - \beta_k) ,$$

with complex roots α_k and β_k.

Appendices: Derivation of product representations IV

- Everything is assembled to write down the product representations:

$$b_1 (W, \theta) = -\mathcal{C}\, a_{2\ell_{\mathrm{max}}} \exp\left(-i\frac{\theta}{2}\right) f(\theta, -\beta),$$

$$b_2 (W, \theta) = -\mathcal{C}\, a_{2\ell_{\mathrm{max}}} \exp\left(i\frac{\theta}{2}\right) f(\theta, \beta),$$

$$b_3 (W, \theta) = \mathcal{C}\, a_{2\ell_{\mathrm{max}}} \exp\left(-i\frac{\theta}{2}\right) f(\theta, -\alpha),$$

$$b_4 (W, \theta) = \mathcal{C}\, a_{2\ell_{\mathrm{max}}} \exp\left(i\frac{\theta}{2}\right) f(\theta, \alpha).$$

using the definition of the root function

$$f(\theta, \alpha) = f(\theta, \alpha_1, \ldots, \alpha_{2\ell_{\mathrm{max}}}) = \frac{\prod_{k=1}^{2\ell_{\mathrm{max}}} \left(\tan\frac{\theta}{2} - \alpha_k\right)}{\left(1 + \tan^2\frac{\theta}{2}\right)^{\ell_{\mathrm{max}}}}.$$

Appendices: Ambiguities of the Group S observables I

- Equivalence for every ℓ_{max}: $\{E_{\ell\pm}, M_{\ell\pm}\} \leftrightarrow \{a_i, b_i\} \leftrightarrow \{\alpha_k, \beta_k\}$.

Appendices: Ambiguities of the Group S observables I

- Equivalence for every ℓ_{max}: $\{E_{\ell\pm}, M_{\ell\pm}\} \leftrightarrow \{a_i, b_i\} \leftrightarrow \{\alpha_k, \beta_k\}$.
- What are the possible ambiguities of the group S observables?

Observable	Root function representation	Type
$I(\theta) = \sigma_0/\rho$	$\frac{I(\pi)}{4}\left(\lvert f(\theta,-\beta)\rvert^2 + \lvert f(\theta,\beta)\rvert^2 + \lvert f(\theta,-\alpha)\rvert^2 + \lvert f(\theta,\alpha)\rvert^2\right)$	
$\check{\Sigma}$	$\frac{I(\pi)}{4}\left(-\lvert f(\theta,-\beta)\rvert^2 - \lvert f(\theta,\beta)\rvert^2 + \lvert f(\theta,-\alpha)\rvert^2 + \lvert f(\theta,\alpha)\rvert^2\right)$	S
$\check{T}$	$\frac{I(\pi)}{4}\left(\lvert f(\theta,-\beta)\rvert^2 - \lvert f(\theta,\beta)\rvert^2 - \lvert f(\theta,-\alpha)\rvert^2 + \lvert f(\theta,\alpha)\rvert^2\right)$	
$\check{P}$	$\frac{I(\pi)}{4}\left(-\lvert f(\theta,-\beta)\rvert^2 + \lvert f(\theta,\beta)\rvert^2 - \lvert f(\theta,-\alpha)\rvert^2 + \lvert f(\theta,\alpha)\rvert^2\right)$	

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$\check{\Sigma}$	$\frac{I(\pi)}{4} \left(- \lvert f(\theta, -\beta)\rvert^2 - \lvert f(\theta,\beta)\rvert^2 + \lvert f(\theta, -\alpha)\rvert^2 + \lvert f(\theta,\alpha)\rvert^2 \right)$	S
$\check{T}$	$\frac{I(\pi)}{4} \left(\lvert f(\theta, -\beta)\rvert^2 - \lvert f(\theta,\beta)\rvert^2 - \lvert f(\theta, -\alpha)\rvert^2 + \lvert f(\theta,\alpha)\rvert^2 \right)$	
$\check{P}$	$\frac{I(\pi)}{4} \left(- \lvert f(\theta, -\beta)\rvert^2 + \lvert f(\theta,\beta)\rvert^2 - \lvert f(\theta, -\alpha)\rvert^2 + \lvert f(\theta,\alpha)\rvert^2 \right)$	

I. Complex conjugation of all roots:

$$\alpha \rightarrow \alpha^* , \quad \beta \rightarrow \beta^*,$$

called the Double Ambiguity transformation.

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$I(\theta) = \sigma_0/\rho$	$\frac{I(\pi)}{4}\left(\lvert f(\theta,-\beta)\rvert^2 + \lvert f(\theta,\beta)\rvert^2 + \lvert f(\theta,-\alpha)\rvert^2 + \lvert f(\theta,\alpha)\rvert^2\right)$	
$\check{\Sigma}$	$\frac{I(\pi)}{4}\left(-\lvert f(\theta,-\beta)\rvert^2 - \lvert f(\theta,\beta)\rvert^2 + \lvert f(\theta,-\alpha)\rvert^2 + \lvert f(\theta,\alpha)\rvert^2\right)$	S
$\check{T}$	$\frac{I(\pi)}{4}\left(\lvert f(\theta,-\beta)\rvert^2 - \lvert f(\theta,\beta)\rvert^2 - \lvert f(\theta,-\alpha)\rvert^2 + \lvert f(\theta,\alpha)\rvert^2\right)$	
$\check{P}$	$\frac{I(\pi)}{4}\left(-\lvert f(\theta,-\beta)\rvert^2 + \lvert f(\theta,\beta)\rvert^2 - \lvert f(\theta,-\alpha)\rvert^2 + \lvert f(\theta,\alpha)\rvert^2\right)$	

I. Complex conjugation of all roots:

$$\alpha \rightarrow \alpha^* , \quad \beta \rightarrow \beta^*,$$

called the Double Ambiguity transformation.

II. Complex of arbitrary subsets of roots α_k and β_k.

Appendices: Ambiguities of the Group S observables II

- One additional condition that has to be fulfilled by a valid ambiguity: $A_{2\ell_{\mathrm{max}}}(t) = \prod_{k=1}^{2\ell_{\mathrm{max}}} (t - \alpha_k)$ and $B_{2\ell_{\mathrm{max}}}(t) = \prod_{k=1}^{2\ell_{\mathrm{max}}} (t - \beta_k)$ combined with $A_{2\ell_{\mathrm{max}}}(t = 0) = B_{2\ell_{\mathrm{max}}}(t = 0)$ yield

$$\prod_{k=1}^{2\ell_{\mathrm{max}}} \alpha_k = \prod_{k=1}^{2\ell_{\mathrm{max}}} \beta_k, \quad \underline{\text{consistency relation}}.$$

  Written down, for the phases φ_k and ψ_k of $\alpha_k = |\alpha_k| e^{i\varphi_k}$ and $\beta_k = |\beta_k| e^{i\psi_k}$

$$\varphi_1 + \ldots + \varphi_{2\ell_{\mathrm{max}}} = \psi_1 + \ldots + \psi_{2\ell_{\mathrm{max}}}.$$

Appendices: Ambiguities of the Group S observables II

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  Written down, for the phases φ_k and ψ_k of $\alpha_k = |\alpha_k| e^{i\varphi_k}$ and $\beta_k = |\beta_k| e^{i\psi_k}$

$$\varphi_1 + \ldots + \varphi_{2\ell_{\max}} = \psi_1 + \ldots + \psi_{2\ell_{\max}}.$$

- The consistency relation is always fulfilled by the Double Ambiguity transformation

$$-\varphi_1 - \ldots - \varphi_{2\ell_{\max}} = -\psi_1 - \ldots - \psi_{2\ell_{\max}}.$$

  Therefore the Double Ambiguity the only ambiguity that can be certainly predicted.

Appendices: Ambiguities of the Group S observables III

- Every other possibility of signs that fulfills the consistency relation

$$\pm\varphi_1 \pm \ldots \pm \varphi_{2\ell_{\max}} = \pm\psi_1 \pm \ldots \pm \psi_{2\ell_{\max}},$$

  is not predictable but is merely a numerical accident. The corresponding ambiguity is called an accidential ambiguity.

Appendices: Ambiguities of the Group S observables III

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Appendices: Ambiguities of the Group S observables III

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Appendices: Ambiguities of the Group S observables III

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- Accidental ambiguities $\tilde{\alpha}_k$ and $\tilde{\beta}_k$ again form pairs by means of the Double Ambiguity transformation: $\tilde{\alpha} \rightarrow \tilde{\alpha}^*$, $\tilde{\beta} \rightarrow \tilde{\beta}^*$.
- In order to remove ambiguities, additional observables from the classes Beam-Target, Beam-Recoil and Target-Recoil are needed.

Appendices: Structure of the Double Ambiguity transformation

The Double Ambiguity transformation $\alpha \rightarrow \alpha^{\ast}$, $\beta \rightarrow \beta^{\ast}$, acts on e.g. the amplitude $b_1 \left(W, \theta \right)$ as (W dependence implicit & $f \left(\theta, \beta^{\ast} \right) = f^{\ast} \left(\theta, \beta \right)$):

$$
\begin{aligned}
b_1^{\prime} \left(\theta \right) \equiv b_1 \left(\theta \right) \left|_{\mathrm{D.A.}} \right. &= - \mathcal{C} \, a_{2 \ell_{\mathrm{max}}} \exp \left(- i \frac{\theta}{2} \right) f \left(\theta, - \beta^{\ast} \right) \\
&= - \mathcal{C} \, a_{2 \ell_{\mathrm{max}}} \exp \left(- i \frac{\theta}{2} \right) f^{\ast} \left(\theta, - \beta \right) \\
&= \exp \left(- i \theta \right) \left(- \mathcal{C} \right) a_{2 \ell_{\mathrm{max}}} \exp \left(+ i \frac{\theta}{2} \right) f^{\ast} \left(\theta, - \beta \right) = \exp \left(- i \theta \right) b_1^{\ast} \left(\theta \right) .
\end{aligned}
$$

Appendices: Structure of the Double Ambiguity transformation

The Double Ambiguity transformation $\alpha \rightarrow \alpha^{*}$, $\beta \rightarrow \beta^{*}$, acts on e.g. the amplitude $b_1 \left(W, \theta \right)$ as (W dependence implicit & $f \left(\theta, \beta^{*} \right) = f^{*} \left(\theta, \beta \right)$):

$$b_1^{\prime} \left(\theta \right) \equiv b_1 \left(\theta \right) |_{\mathrm{D.A.}} = - \mathcal{C} \, a_{2 \ell_{\mathrm{max}}} \exp \left(- i \frac{\theta}{2} \right) f \left(\theta, - \beta^{*} \right)$$

$$= - \mathcal{C} \, a_{2 \ell_{\mathrm{max}}} \exp \left(- i \frac{\theta}{2} \right) f^{*} \left(\theta, - \beta \right)$$

$$= \exp \left(- i \theta \right) \left(- \mathcal{C} \right) a_{2 \ell_{\mathrm{max}}} \exp \left(+ i \frac{\theta}{2} \right) f^{*} \left(\theta, - \beta \right) = \exp \left(- i \theta \right) b_1^{*} \left(\theta \right) .$$

More generally, it is a θ dependent antilinear transformation

$$b_i \left(\theta \right) \longrightarrow b_i^{\prime} \left(\theta \right) = \sum_{j} \mathcal{A}_{ij} \left(\theta \right) b_j^{*}$$

with transformation matrix:

$$\mathcal{A} \left(\theta \right) = \mathrm{diag} \left(\exp \left(- i \theta \right) , \, \exp \left(i \theta \right) , \, \exp \left(- i \theta \right) , \, \exp \left(i \theta \right) \right) .$$

Appendices: Double Ambiguity acting on observables I

- Check for every observable

$$\check{\Omega}^{\alpha}\left(W,\theta\right) = \frac{1}{2}\left\langle b\right|\tilde{\Gamma}^{\alpha}\left|b\right\rangle ,$$

whether or not the condition that has to be valid in order to identify $\mathcal{A}\left(\theta\right)$ as an antilinear ambiguity of the observable:

$$\left(\mathcal{A}^{\dagger}\left(\theta\right)\tilde{\Gamma}^{\alpha}\mathcal{A}\left(\theta\right)\right)^{T} = \tilde{\Gamma}^{\alpha},$$

is fulfilled. [Chiang/Tabakin(1996)]

Appendices: Double Ambiguity acting on observables I

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is fulfilled. [Chiang/Tabakin(1996)]

- The Double Ambiguity can be expanded into $\tilde{\Gamma}$ matrices as:

$$\mathcal{A}\left(\theta\right) = \begin{bmatrix} e^{-i\theta} & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 \\ 0 & 0 & e^{-i\theta} & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{bmatrix} = \cos(\theta)\tilde{\Gamma}^{1} + i\sin(\theta)\tilde{\Gamma}^{12}.$$

Appendices: Double Ambiguity acting on observables II

Group S Observable	α	$\tilde{\Gamma}^{\alpha}$		$\left(\mathcal{A}^{\dagger}(\theta)\,\tilde{\Gamma}^{\alpha}\mathcal{A}(\theta)\right)^{T}$
$I(\theta) = \sigma_0/\rho$	1	$\tilde{\Gamma}^1$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	
$\check{\Sigma}$	4	$\tilde{\Gamma}^4$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	
$\check{T}$	10	$\tilde{\Gamma}^{10}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	
$\check{P}$	12	$\tilde{\Gamma}^{12}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	

Appendices: Double Ambiguity acting on observables II

Group S Observable	α		$\tilde{\Gamma}^{\alpha}$	$\left(\mathcal{A}^{\dagger}(\theta)\,\tilde{\Gamma}^{\alpha}\mathcal{A}(\theta)\right)^{T}$
$I(\theta) = \sigma_0/\rho$	1	$\tilde{\Gamma}^{1}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\check{\Sigma}$	4	$\tilde{\Gamma}^{4}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
$\check{T}$	10	$\tilde{\Gamma}^{10}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
$\check{P}$	12	$\tilde{\Gamma}^{12}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Appendices: Double Ambiguity acting on observables II

Group S Observable	α		$\tilde{\Gamma}^{\alpha}$	$\left(\mathcal{A}^{\dagger}(\theta)\,\tilde{\Gamma}^{\alpha}\mathcal{A}(\theta)\right)^{T}$	
$I(\theta) = \sigma_0/\rho$	1	$\tilde{\Gamma}^1$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\tilde{\Gamma}^1$
$\check{\Sigma}$	4	$\tilde{\Gamma}^4$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$\tilde{\Gamma}^4$
$\check{T}$	10	$\tilde{\Gamma}^{10}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$\tilde{\Gamma}^{10}$
$\check{P}$	12	$\tilde{\Gamma}^{12}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\tilde{\Gamma}^{12}$

Appendices: Double Ambiguity acting on observables II

Beam-Target Observable	α	$\tilde{\Gamma}^{\alpha}$		$\left(\mathcal{A}^{\dagger}(\theta)\,\tilde{\Gamma}^{\alpha}\mathcal{A}(\theta)\right)^{T}$
$\check{G}$	3	$\tilde{\Gamma}^{3}$	$\begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}$	
$\check{H}$	5	$\tilde{\Gamma}^{5}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$	
$\check{E}$	9	$\tilde{\Gamma}^{9}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	
$\check{F}$	11	$\tilde{\Gamma}^{11}$	$\begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}$	

Appendices: Double Ambiguity acting on observables II

Beam-Target Observable	α	$\tilde{\Gamma}^{\alpha}$		$\left(\mathcal{A}^{\dagger} \left(\theta \right) \tilde{\Gamma}^{\alpha} \mathcal{A} \left(\theta \right) \right)^{T}$
$\check{G}$	3	$\tilde{\Gamma}^{3}$	$\begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$
$\check{H}$	5	$\tilde{\Gamma}^{5}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$
$\check{E}$	9	$\tilde{\Gamma}^{9}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$
$\check{F}$	11	$\tilde{\Gamma}^{11}$	$\begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$

Appendices: Double Ambiguity acting on observables II

Beam-Target Observable	α		$\tilde{\Gamma}^{\alpha}$	$\left(\mathcal{A}^{\dagger}(\theta)\,\tilde{\Gamma}^{\alpha}\mathcal{A}(\theta)\right)^{T}$	
$\check{G}$	3	$\tilde{\Gamma}^{3}$	$\begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$	$-\tilde{\Gamma}^{3}$
$\check{H}$	5	$\tilde{\Gamma}^{5}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$	$\tilde{\Gamma}^{5}$
$\check{E}$	9	$\tilde{\Gamma}^{9}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\tilde{\Gamma}^{9}$
$\check{F}$	11	$\tilde{\Gamma}^{11}$	$\begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$	$-\tilde{\Gamma}^{11}$

Appendices: Double Ambiguity acting on observables II

Beam-Recoil Observable	α	$\tilde{\Gamma}^{\alpha}$	$\left(\mathcal{A}^{\dagger}(\theta)\,\tilde{\Gamma}^{\alpha}\mathcal{A}(\theta)\right)^{T}$
$\check{O}_{x'}$	14	$\tilde{\Gamma}^{14}$ $\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$	
$\check{O}_{z'}$	7	$\tilde{\Gamma}^{7}$ $\begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$	
$\check{C}_{x'}$	16	$\tilde{\Gamma}^{16}$ $\begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$	
$\check{C}_{z'}$	2	$\tilde{\Gamma}^{2}$ $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	

Appendices: Double Ambiguity acting on observables II

Beam-Recoil Observable	α		$\tilde{\Gamma}^{\alpha}$	$\left(\mathcal{A}^{\dagger}(\theta)\,\tilde{\Gamma}^{\alpha}\mathcal{A}(\theta)\right)^{T}$
$\check{O}_{x'}$	14	$\tilde{\Gamma}^{14}$	$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & -e^{-i2\theta} \\ 0 & 0 & e^{i2\theta} & 0 \\ 0 & e^{-i2\theta} & 0 & 0 \\ -e^{-i2\theta} & 0 & 0 & 0 \end{bmatrix}$
$\check{O}_{z'}$	7	$\tilde{\Gamma}^{7}$	$\begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & ie^{-i2\theta} \\ 0 & 0 & ie^{i2\theta} & 0 \\ 0 & -ie^{i2\theta} & 0 & 0 \\ -ie^{-i2\theta} & 0 & 0 & 0 \end{bmatrix}$
$\check{C}_{x'}$	16	$\tilde{\Gamma}^{16}$	$\begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & -ie^{i2\theta} \\ 0 & 0 & ie^{i2\theta} & 0 \\ 0 & -ie^{i2\theta} & 0 & 0 \\ ie^{i2\theta} & 0 & 0 & 0 \end{bmatrix}$
$\check{C}_{z'}$	2	$\tilde{\Gamma}^{2}$	$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & e^{-i2\theta} \\ 0 & 0 & e^{i2\theta} & 0 \\ 0 & e^{-i2\theta} & 0 & 0 \\ e^{i2\theta} & 0 & 0 & 0 \end{bmatrix}$

Appendices: Double Ambiguity acting on observables II

Beam-Recoil Observable	α	$\tilde{\Gamma}^{\alpha}$		$\left(\mathcal{A}^{\dagger}(\theta)\,\tilde{\Gamma}^{\alpha}\mathcal{A}(\theta)\right)^{T}$
$\check{O}_{x'}$	14	$\tilde{\Gamma}^{14}$	$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$	$\cos(2\theta)\tilde{\Gamma}^{14} - \sin(2\theta)\tilde{\Gamma}^{7}$
$\check{O}_{z'}$	7	$\tilde{\Gamma}^{7}$	$\begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$	$-\cos(2\theta)\tilde{\Gamma}^{7} - \sin(2\theta)\tilde{\Gamma}^{14}$
$\check{C}_{x'}$	16	$\tilde{\Gamma}^{16}$	$\begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$	$-\cos(2\theta)\tilde{\Gamma}^{16} - \sin(2\theta)\tilde{\Gamma}^{2}$
$\check{C}_{z'}$	2	$\tilde{\Gamma}^{2}$	$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\cos(2\theta)\tilde{\Gamma}^{2} - \sin(2\theta)\tilde{\Gamma}^{16}$

Appendices: Double Ambiguity acting on observables II

Target-Recoil Observable	α	$\tilde{\Gamma}^{\alpha}$	$\left(\mathcal{A}^{\dagger}(\theta)\,\tilde{\Gamma}^{\alpha}\mathcal{A}(\theta)\right)^{T}$
$\check{T}_{x'}$	6	$\tilde{\Gamma}^{6}$ $\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	
$\check{T}_{z'}$	13	$\tilde{\Gamma}^{13}$ $\begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$	
$\check{L}_{x'}$	8	$\tilde{\Gamma}^{8}$ $\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$	
$\check{L}_{z'}$	15	$\tilde{\Gamma}^{15}$ $\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	

Appendices: Double Ambiguity acting on observables II

Target-Recoil Observable	α	$\tilde{\Gamma}^{\alpha}$		$\left(\mathcal{A}^{\dagger}(\theta)\,\tilde{\Gamma}^{\alpha}\mathcal{A}(\theta)\right)^{T}$
$\check{T}_{x'}$	6	$\tilde{\Gamma}^{6}$	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -e^{-i2\theta} & 0 & 0 \\ -e^{-i2\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i2\theta} \\ 0 & 0 & e^{i2\theta} & 0 \end{bmatrix}$
$\check{T}_{z'}$	13	$\tilde{\Gamma}^{13}$	$\begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -ie^{i2\theta} & 0 & 0 \\ ie^{i2\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & ie^{-i2\theta} \\ 0 & 0 & -ie^{-i2\theta} & 0 \end{bmatrix}$
$\check{L}_{x'}$	8	$\tilde{\Gamma}^{8}$	$\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & ie^{-i2\theta} & 0 & 0 \\ -ie^{-i2\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & ie^{-i2\theta} \\ 0 & 0 & -ie^{-i2\theta} & 0 \end{bmatrix}$
$\check{L}_{z'}$	15	$\tilde{\Gamma}^{15}$	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -e^{-i2\theta} & 0 & 0 \\ -e^{-i2\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{-i2\theta} \\ 0 & 0 & -e^{-i2\theta} & 0 \end{bmatrix}$

Appendices: Double Ambiguity acting on observables II

Target-Recoil Observable	α	$\tilde{\Gamma}^{\alpha}$		$\left(\mathcal{A}^{\dagger}(\theta)\,\tilde{\Gamma}^{\alpha}\mathcal{A}(\theta)\right)^{T}$
$\check{T}_{x'}$	6	$\tilde{\Gamma}^{6}$	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\cos(2\theta)\tilde{\Gamma}^{6} + \sin(2\theta)\tilde{\Gamma}^{13}$
$\check{T}_{z'}$	13	$\tilde{\Gamma}^{13}$	$\begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$	$-\cos(2\theta)\tilde{\Gamma}^{13} + \sin(2\theta)\tilde{\Gamma}^{6}$
$\check{L}_{x'}$	8	$\tilde{\Gamma}^{8}$	$\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$	$-\cos(2\theta)\tilde{\Gamma}^{8} - \sin(2\theta)\tilde{\Gamma}^{15}$
$\check{L}_{z'}$	15	$\tilde{\Gamma}^{15}$	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	$\cos(2\theta)\tilde{\Gamma}^{15} - \sin(2\theta)\tilde{\Gamma}^{8}$