Introducing Pietarinen expansion method into single-channel (!) pole extraction problem

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Motivation and justification

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Poles are finally established as the ultimate resonance criterion

1. ***Conclusions of ATHOS 2012, ATHOS2013***

2. ***Recent change in PDG attitude***

Immediate problem:

It is a common knowledge how to extract Breit-Wigner parameters from experimental data,

However, it is rather obscure how to do it with poles

We know how to extract Breit-Wigner parameters from experiment because they are defined on the real axes.

(see Camogli Michel)

But, how do we extract pole parameters from experiment because we have to go to the complex energy plane?

The usual answer was:

1. ***Do it globally***

One first has to make a model which fits the data, SOLVE IT, and obtain an explicit analytic function in the full complex energy plane. Second, one has to look for the complex poles of the obtained analytic functions.

2. ***Do it locally***

Speed plot, expansions in power series, etc

Taylor expansion

PHYSICAL REVIEW D **90,** 097901 (2014)

Precise determination of resonance pole parameters through Padé approximants

Pere Masjuan,[1,*] Jacobo Ruiz de Elvira,[2,†] and Juan José Sanz-Cillero[3,‡]

Let us consider a function $F(x)$, analytical in a disk $B_\delta(x_0)$. Then, the Taylor expansion

$$\mathcal{P}_N(x, x_0) = \sum_{n=0}^{N} a_n (x - x_0)^n \,, \qquad (1)$$

converges to $F(x)$ in $B_\delta(x_0)$ for $N \to \infty$, with derivatives given by $a_n = F^{(n)}(x_0)/n!$.

The scenario changes, however, when the function $F(x)$ is not analytical anymore, for example when it has a single pole at $x = x_p$ inside the disk $B_\delta(x_0)$. In this case, the Taylor series does not converge any more, so we need a different procedure to extract information about the function and its derivatives.

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A special case of interest for the present work is Montessus de Ballore's theorem [6, 17, 18]. Montessus' theorem states that when the amplitude $F(x)$ is analytical inside the disk $B_\delta(x_0)$ except for a single pole at $x = x_p$ the sequence of one-pole Padé Approximants $P_1^N(x, x_0)$ around x_0,

$$P_1^N(x, x_0) = \sum_{k=0}^{N-1} a_k (x - x_0)^k + \frac{a_N (x - x_0)^N}{1 - \frac{a_{N+1}}{a_N}(x - x_0)}, \quad (3)$$

converges to $F(x)$ in any compact subset of the disk excluding the pole x_p, i.e,

$$\lim_{N\to\infty} P_1^N(x, x_0) = F(x). \quad (4)$$

Regularization method

PHYSICAL REVIEW D **77**, 116007 (2008)

Resolution of the multichannel anomaly in the extraction of S-matrix resonance-pole parameters

Saša Ceci,[1,2,*] Jugoslav Stahov,[3,4] Alfred Švarc,[1] Shon Watson,[3] and Branimir Zauner[1]

The function $T(z)$ with a simple pole at μ is regularized by multiplying it with a simple zero at μ

$$f(z) = (\mu - z)T(z). \tag{8}$$

From this definition and Eq. (7), it is evident that the value of $f(\mu)$ is equal to the residue r of $T(z)$ at point μ. As we have the access to the function values on real axis only, the Taylor expansion of f is performed about some real x to give the value (residue) at the pole μ (where background is highly suppressed)

$$f(\mu) = \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!}(\mu - x)^n + R_N(x, \mu). \tag{9}$$

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The expansion is explicitly written to the order N and the remainder is designated by $R_N(x, \mu)$. Using the mathematical induction one can show that the Nth derivative of $f(x)$, given by Eq. (8), is

$$f^{(n)}(x) = (\mu - x)T^{(n)}(x) - nT^{(n-1)}(x). \qquad (10)$$

Insertion of this derivative into the Taylor expansion conveniently cancels all consecutive terms in the sum, except the last one

$$f(\mu) = \frac{T^{(N)}(x)}{N!}(\mu - x)^{(N+1)} + R_N(x, \mu), \qquad (11)$$

$\mu = a + ib$

function residue $|f(\mu)|$

$$\frac{(a-x)^2 + b^2}{\sqrt[N+1]{|f(\mu)|^2}} = \sqrt[N+1]{\frac{(N!)^2}{|T^{(N)}(x)|^2}}. \qquad (13)$$

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In both cases we have   n-TH DERIVATIVE of the function

PROBLEMS for local solutions !

Direct problems for global solutions:

- ***Many models***
- ***Complicated and different analytic structure***
- ***Elaborated method for solving the problem***
- ***SINGLE USER RESULTS***

In Camogli 2012, during „coffee-break conversation" I have claimed that extracting poles from theoretical and even from experimental data should in principle be possible, and I have promised to try to propose a simple method.

Now I am fulfilling this promise.

Is it possible to create universal approach, usable for everyone, and above all REPRODUCIBLE?

I have tryed to do it starting from very general principles:

1. ***Analyticity***
2. ***Unitarity***

Idea: ***TRADING ADVANTAGES***

GLOBALITY FOR SIMPLICITY

THEORETICAL MODELS

*If you create a model, the advantage is that your solution is absolutely **global,** valid in the full complex energy plane (all Rieman sheets). The drawback is that the solution is **complicated,** pole positions are usually energy dependent otherwise you cannot ensure simple physical requirements like absence of the poles on the first, physical Riemann sheet, Schwartz reflection principle, etc. It is complicated and demanding to solve it.*

WE PROPOSE

Construct an analytic function NOT in the full complex energy plane, but CLOSE to the real axes in the area of dominant nucleon resonances, which is fitting the data by using

LAURENT EXPANSION.

Why Laurent's decomposition?

- *It is a unique representation of the complex analytic function on a dense set in terms of pole parts and regular background*
- *It explicitly seperates pole terms from regular part*
- *It has constant pole parameters*
- *It is not a representation in the full complex energy plane, but has its well defined area of convergence*

IMPORTANT TO UNDERSTAND:

It is not an expansion in pole positions with constant coefficients (as some referees reproached), because it is defined only in a part of the complex energy plane.

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Expansion of the T-matrix in terms of constant coefficients

$$T(\omega) \approx \sum_{i=1}^{k} \frac{x_i \, \Gamma_i/2}{\omega - M_i - \imath \Gamma_i/2} + B(\omega)$$

$$x_i, \; M_i, \; \Gamma_i, \omega \in \mathbb{R}. \qquad (4)$$

cannot be valid in principle.

Namely, poles with constant coefficients have poles on ALL physical sheets, and that violates common sense because only bound states are allowed to be located on the physical sheet.

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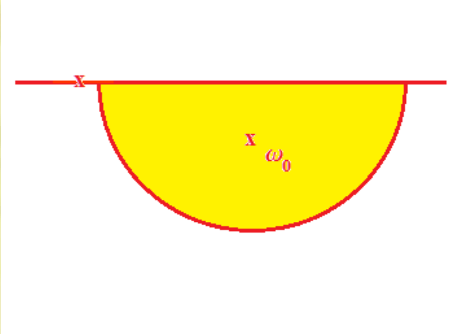
The only way how to accomodate both, requirements of absence of poles on the physical sheet, and Schwartz principle requires that pole positions become energy dependent:

$$\frac{\Gamma(\omega)e^{\imath\phi}}{\omega - M(\omega) - \imath K(\omega)\Gamma(\omega)}$$

However, even this function has its Laurent decomposition

$$\frac{\Gamma(\omega)e^{\imath\phi}}{\omega - M(\omega) - \imath K(\omega)\Gamma(\omega)} \equiv \frac{a_{-1}}{\omega - \omega_0} + \sum_{n=0}^{\infty} a_n(\omega - \omega_0)^n. \tag{5}$$

But it is valid only in the part of the complex energy plane



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1. Analyticity

Analyticity is introduced via generalized Laurent's decomposition
(Mittag-Leffler theorem)

However, the functions we meet and analyze in reality may and do contain more than one pole for $\omega \neq \omega_0$. So if we iterate this procedure using Mittag-Leffler theorem [4] which says that a meromorphic function can be expressed in terms of its poles and associated residues combined with additional entire function, we can without loss of generality write down the generalized Laurent expansion for the function with k poles:

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Assumption:

- ***We are working with first order poles so all negative powers in Laurent's expansion lower than***

  $n < -1$

  are suppressed

Now, we have two parts¸of Laurent's decomposition:

1. ***Poles***
2. ***Regular part***

Idea: TO MIMICK THE PROCEDURE FOR BREIT-WIGNER CASE

Bw:

$$T = \frac{x\,\frac{\Gamma}{2}}{M - w - i\,\frac{\Gamma}{2}} + Bg(w)$$

With Laurent's decompositions for simple poles

$$T(\omega) = \frac{(a_R + i\;a_I)_{-1}}{\omega_0 - \omega} + \sum_{n=0}^{\infty} a_n (\omega_0 - \omega)^n$$

where

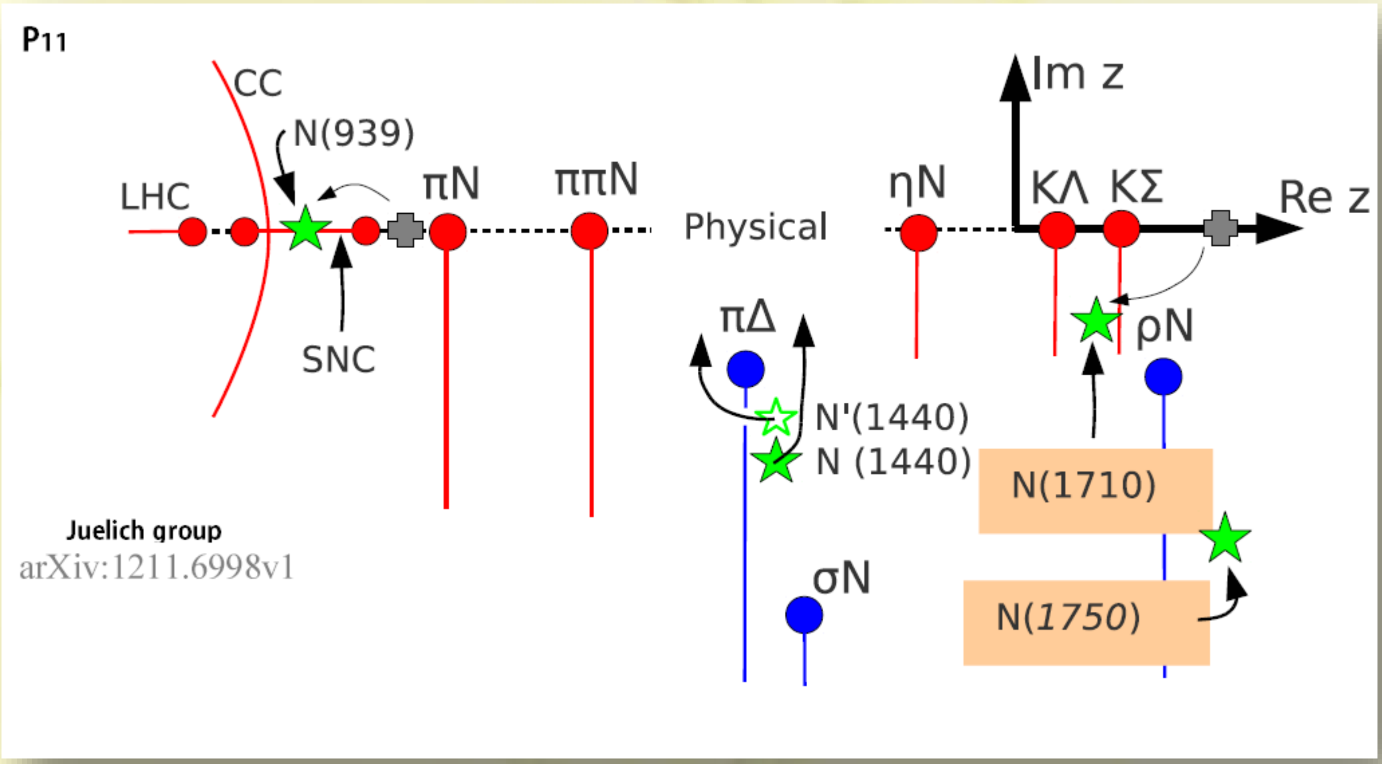
regular function

$$\sum_{n=0}^{\infty} a_n (\omega_0 - \omega)^n = \tilde{B}(\omega)$$

The problem is how to determine regular function $\tilde{B}(w)$.

What do we know about it?

We know it's analytic structure for each partial wave!



We do not know its EXPLICT analytic form!

So, instead of „guessing” its exact form by using model assumptions we

EXPAND IT IN FASTLY CONVERGENT POWER SERIES OF PIETARINEN („Z”) FUNCTIONS WITH WELL KNOWN BRANCH-POINTS!

Original idea:

1. *S. Ciulli and J. Fischer in Nucl. Phys. 24, 465 (1961)*
2. *I. Ciulli, S. Ciulli, and J. Fisher, Nuovo Cimento 23, 1129 (1962).*

Convergence proven in:

1. *S. Ciulli and J. Fischer in Nucl. Phys. 24, 465 (1961)*
2. *Detailed proof in I. Caprini and J. Fischer: "Expansion functions in perturbative QCD and the determination of α_s", Phys.Rev. D84 (2011) 054019,*

Applied in πN scattering on the level of invariant

PENA
IN

1. *E. Pietarinen, Nuovo Cimento Soc. Ital. Fis. 12A, 522 (1972).*
2. *Hoehler – Landolt Boernstein „BIBLE” (1983)*

i) Fixed-*t* analysis

The analysis would be too complicated if one would insist on working with dispersion integrals. It is possible only by the use of PIETARINENs expansion of the invariant amplitudes in terms of functions which have the correct analytic properties (Sect. A.6.3.4). The coefficients are determined from fits to the data, using the fixed-*s* solution as a constraint.

What is Pitarinen's expansion?

In principle, in mathematical language, it is " ...a conformal mapping which maps the physical sheet of the ω-plane onto the interior of the unit circle in the Z-plane..."

In practice this means:

If $F(\omega)$ is a general, unknown analytic function having a cut starting at $\omega = x_P$, then it can be represented in a power series of Pietarinen functions in the following way:

$$F(\omega) = \sum_{n=0}^{N} c_n \, Z(\omega)^n, \qquad \omega \in \mathbb{C}$$

$$Z(\omega) = \frac{\alpha - \sqrt{x_P - \omega}}{\alpha + \sqrt{x_P - \omega}}, \quad c_n, x_P, \alpha \in \mathbb{R}, \qquad (3)$$

with the α and c_n being tuning parameter and coefficients of Pietarinen function $Z(\omega)$ respectively.

Or in another words, Pietarinen functions Z(ω) are a complet set of functions for an arbitrary function F(ω) which HAS A BRANCH POINT AT xP !

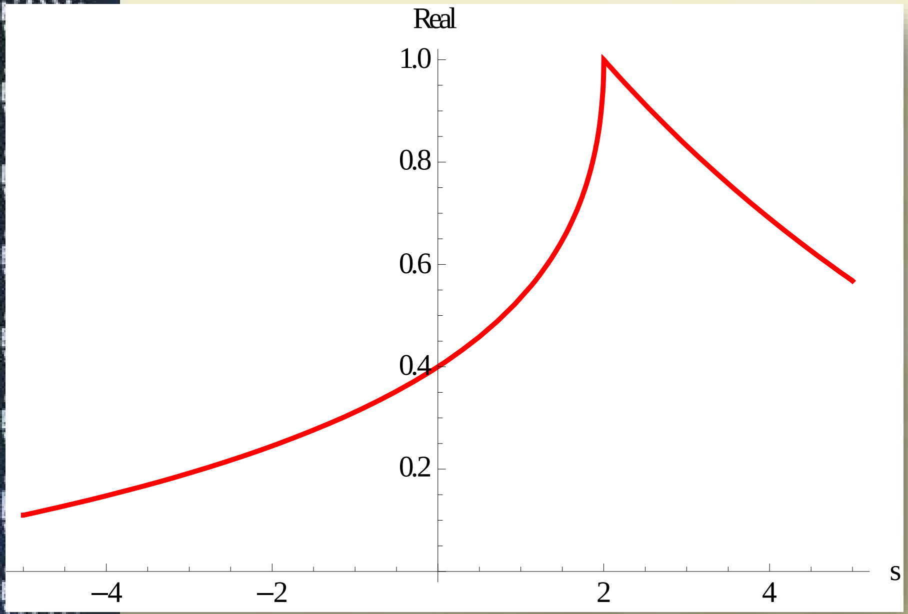
Observe:

Pietarinen functions do not form a complete set of functions for any function, but only for the function having a well defined branch point.

Illustration:

Powes series for Z(ω) = $\dfrac{3.3-\sqrt{2-\omega}}{3.3+\sqrt{2-\omega}}$

Z(ω)



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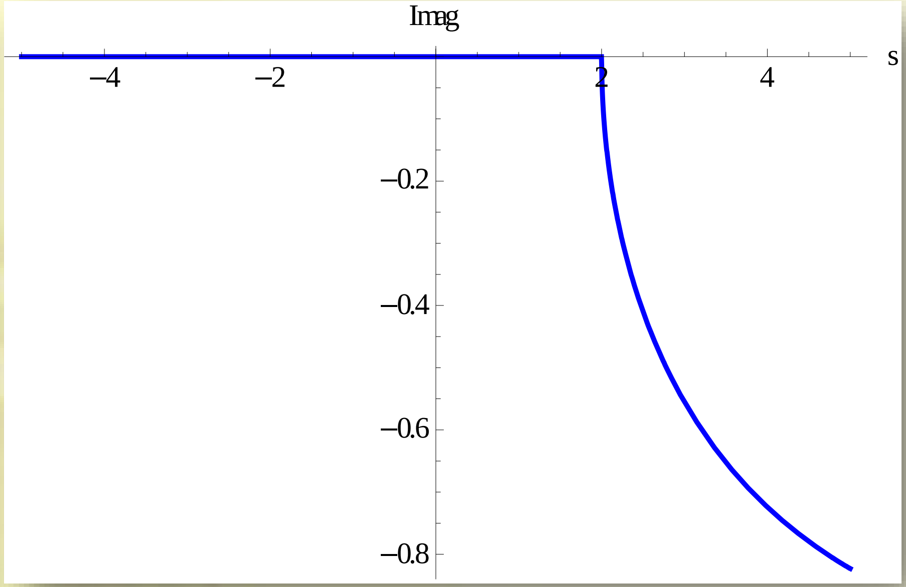

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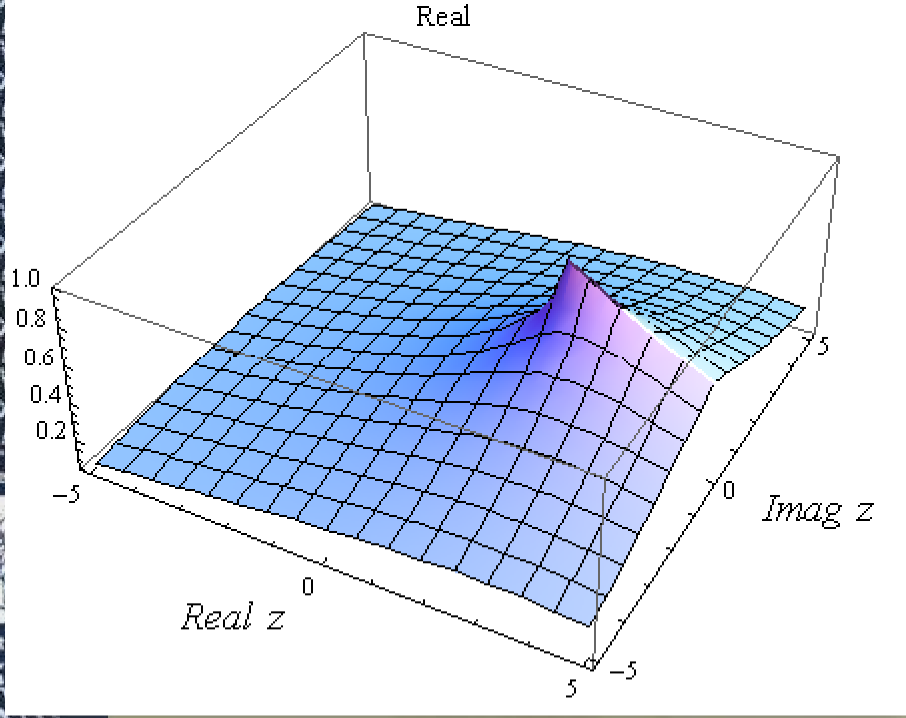

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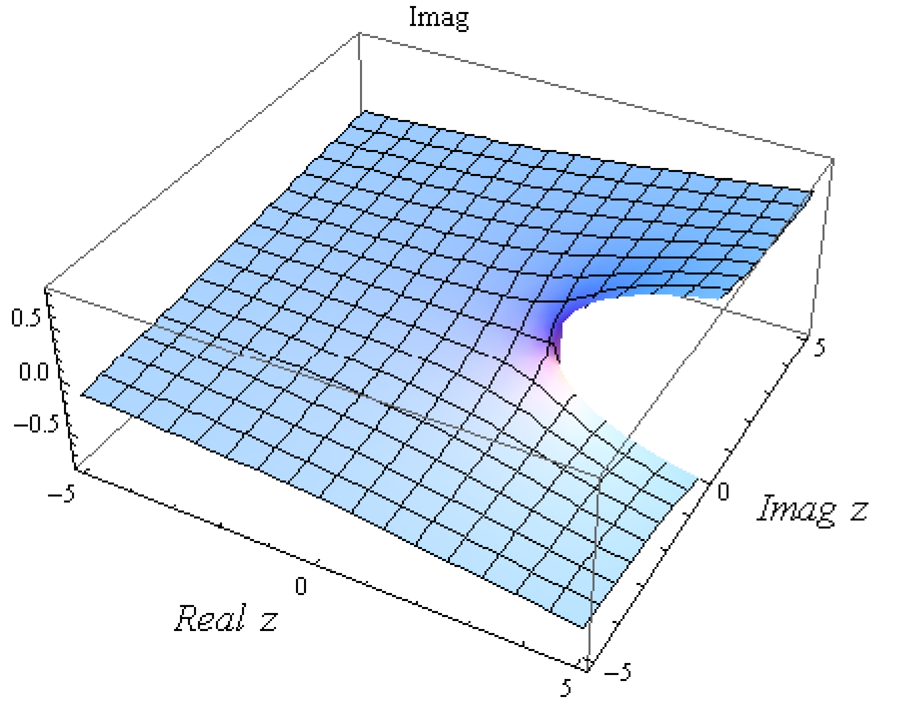

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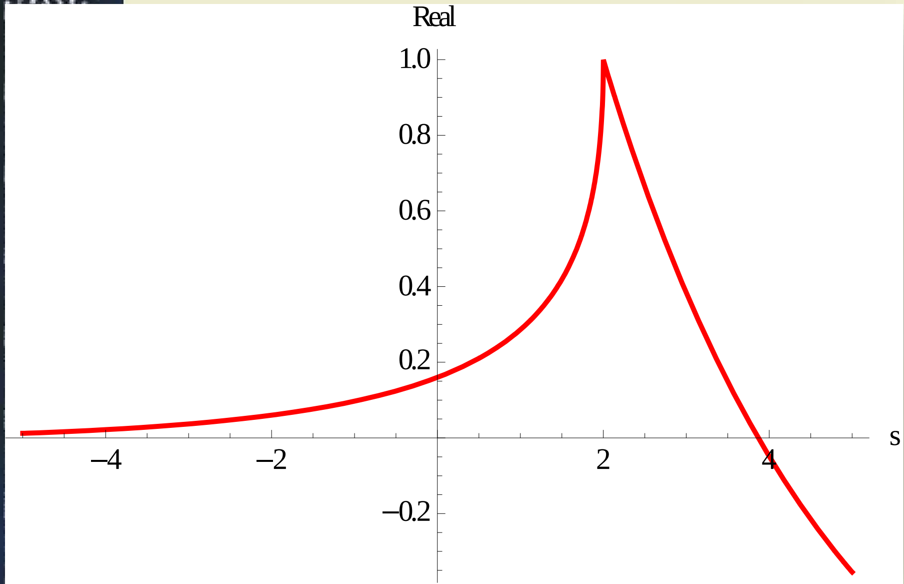
$Z(\omega)^2$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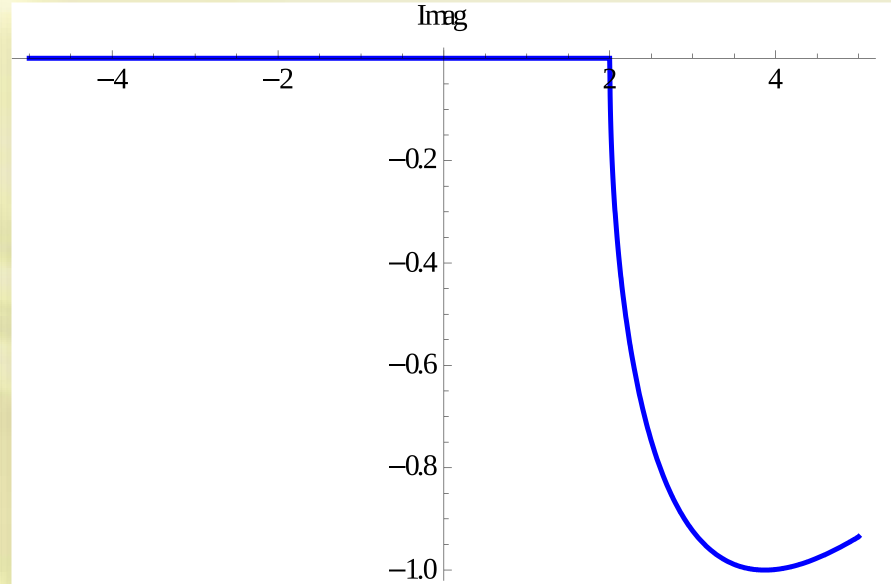
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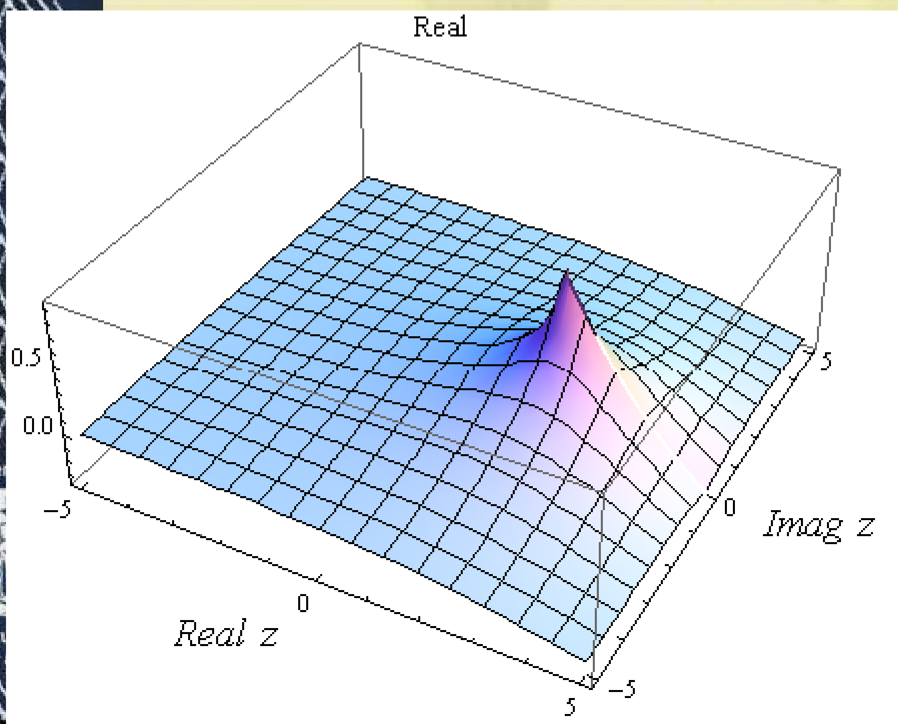
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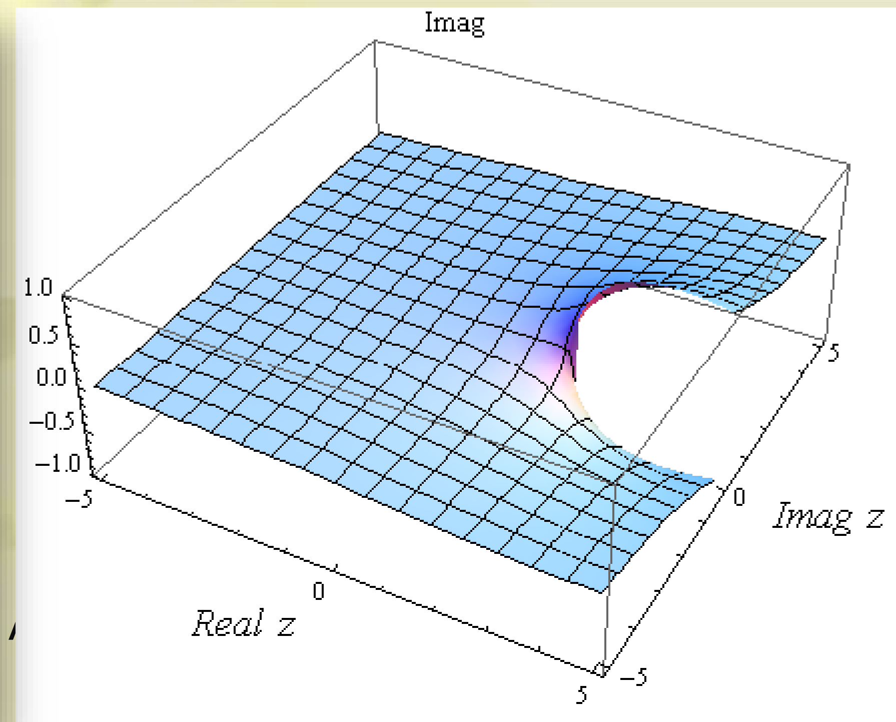
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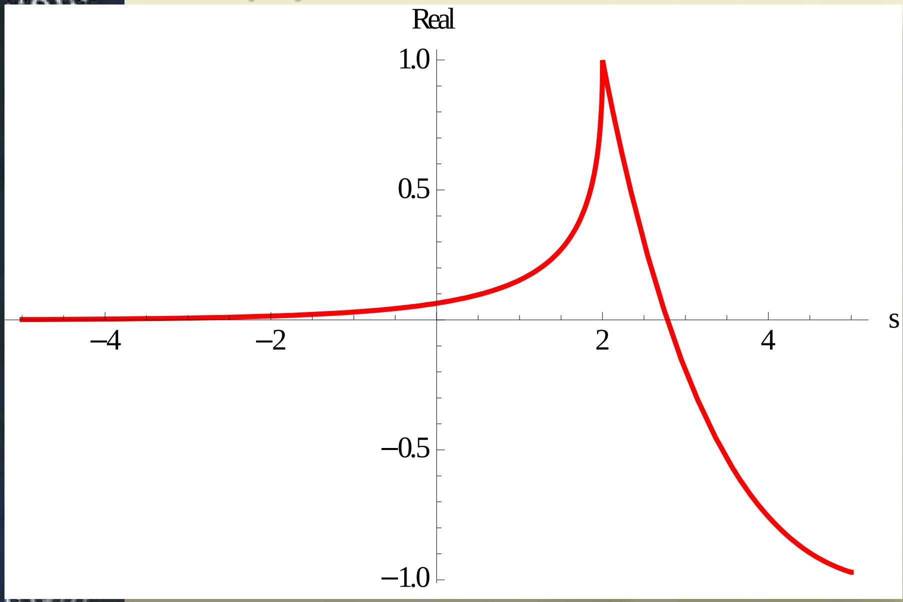
$Z(\omega)^3$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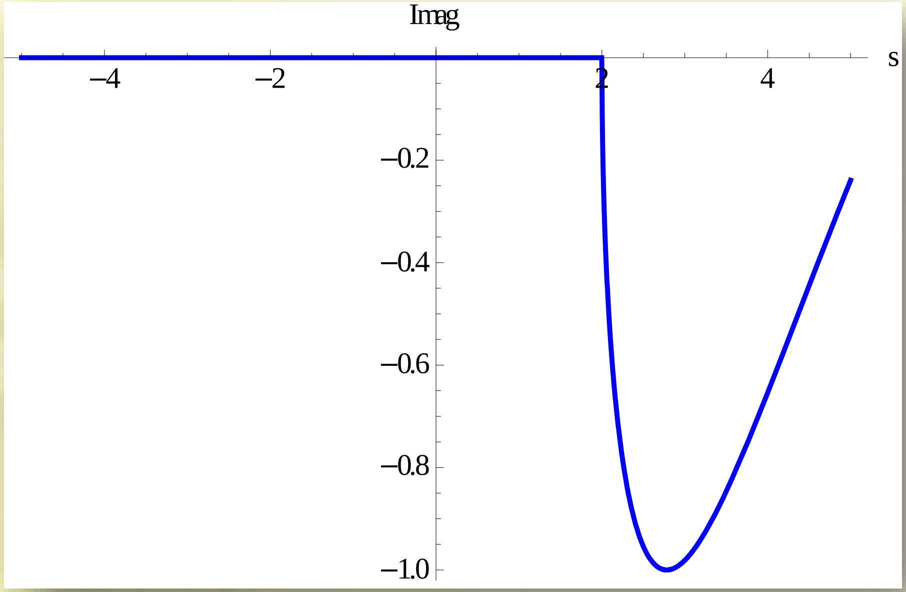
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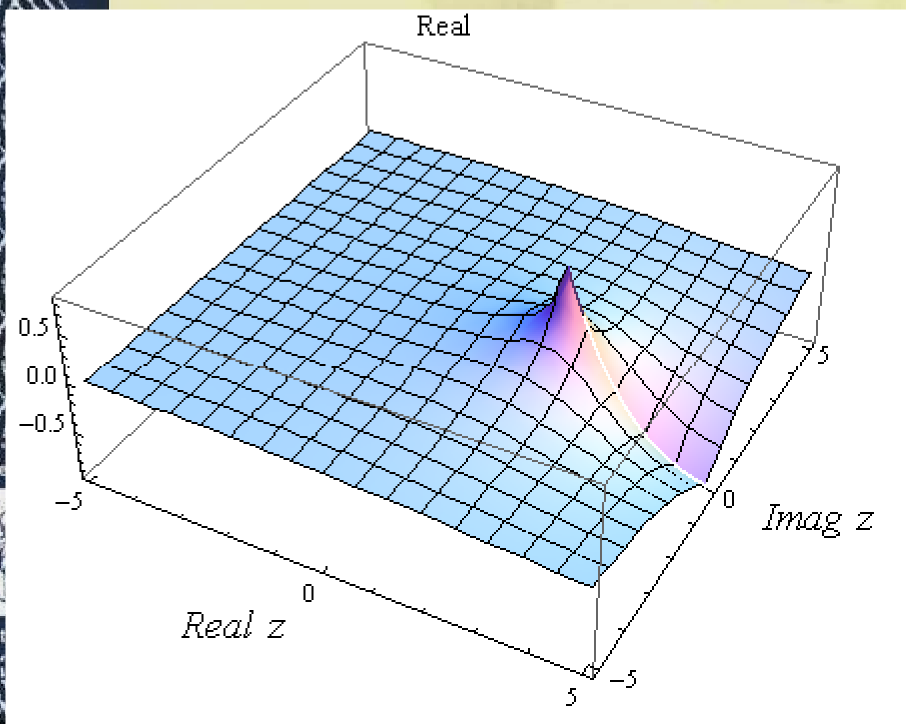
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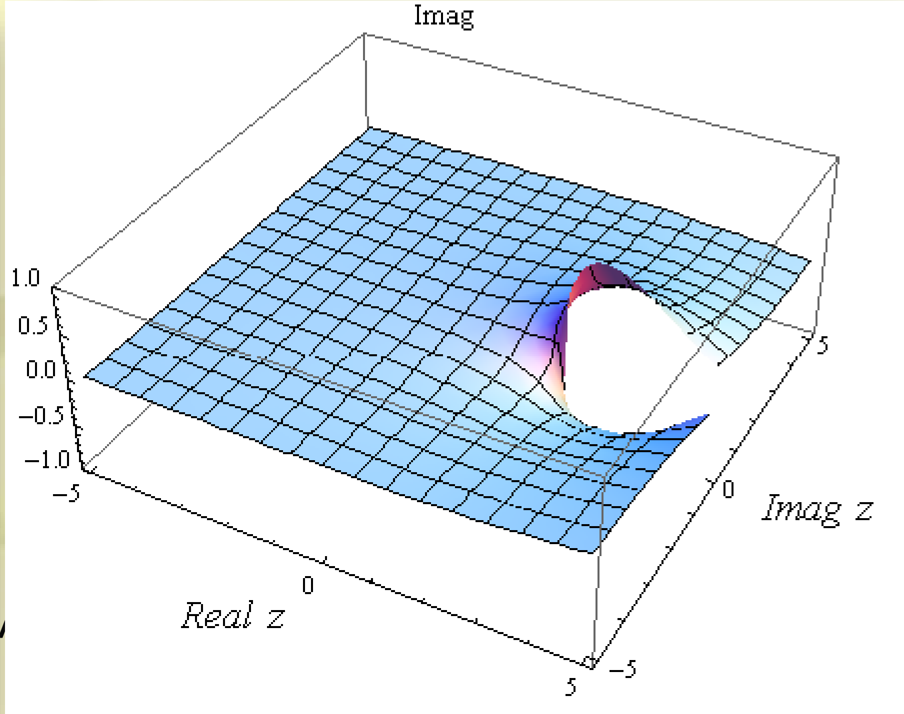
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Important!

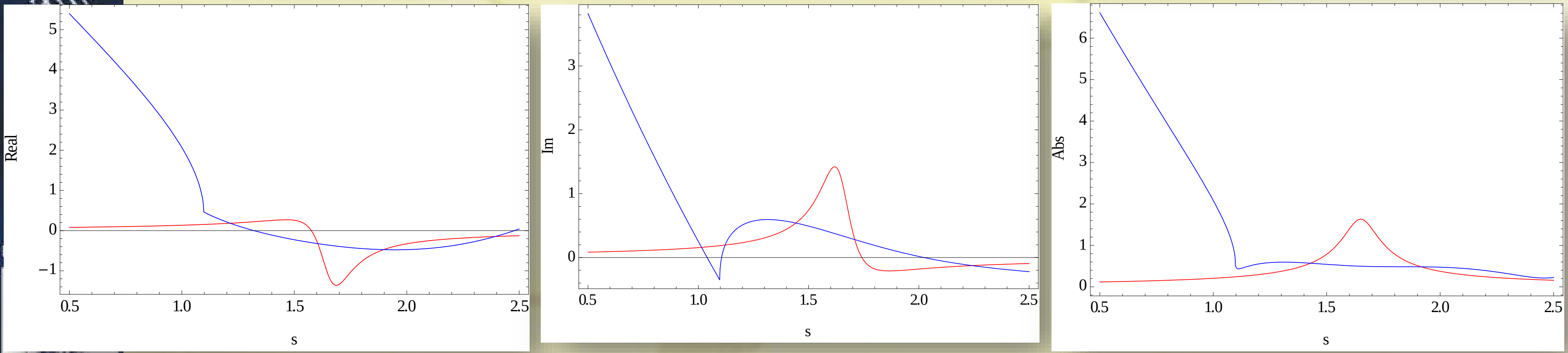
A resonance CANNOT be well described by Pietarinen series.

$$BW[s\_] := r1 \Big/ \left(M1 - s - i \frac{\Gamma1}{2} \right)$$

```
ZI[x_] := (α - √(xP - x)) / (α + √(xP - x));
WI[x_] := (β - √(xQ - x)) / (β + √(xQ - x));


FPietfit[x_] := c0 * ZI[x]^0 + c1 * ZI[x]^1 + c2 * ZI[x]^2 + c3 * ZI[x]^3 + c4 * ZI[x]^4 +
    c5 * ZI[x]^5 - d0 * WI[x]^0 - d1 * WI[x]^1 - d2 * WI[x]^2 - d3 * WI[x]^3 - d4 * WI[x]^4 -
    d5 * WI[x]^5;
```

$xP \to -4.93028,\ xQ \to 1.09731$



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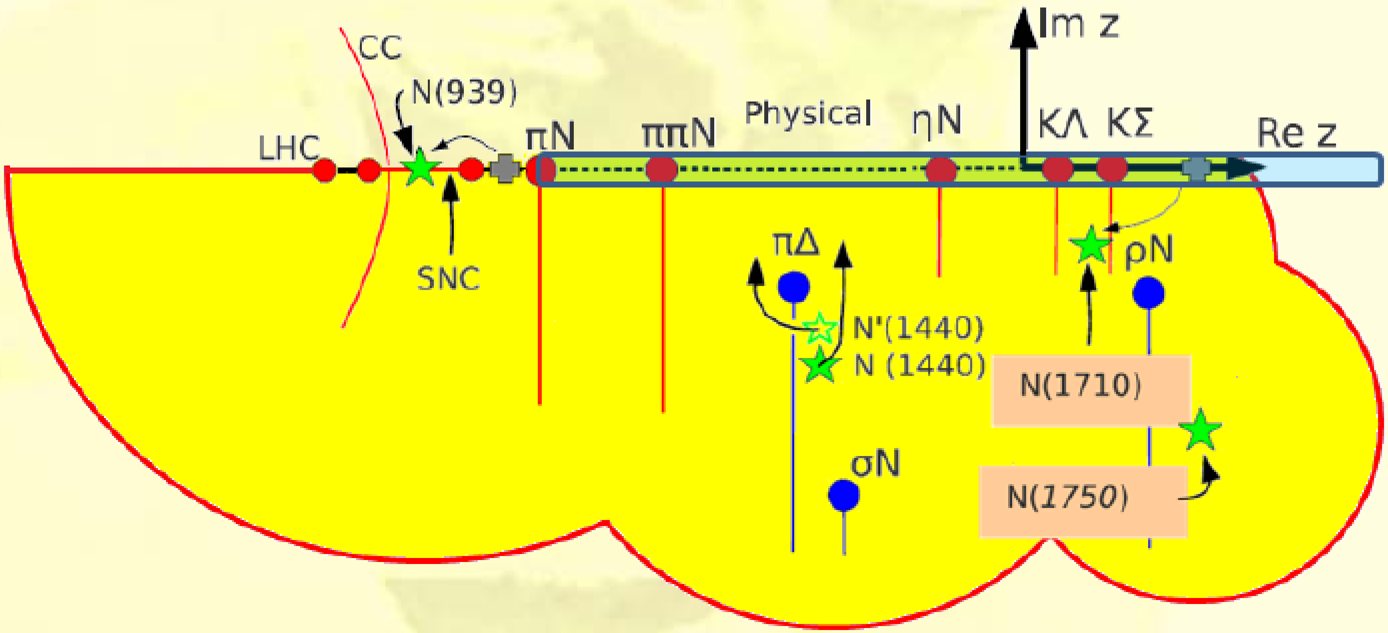
Courtesy of Lothar Tiator

Finally, the area of convergence for Laurent expansion of P11 partial wave

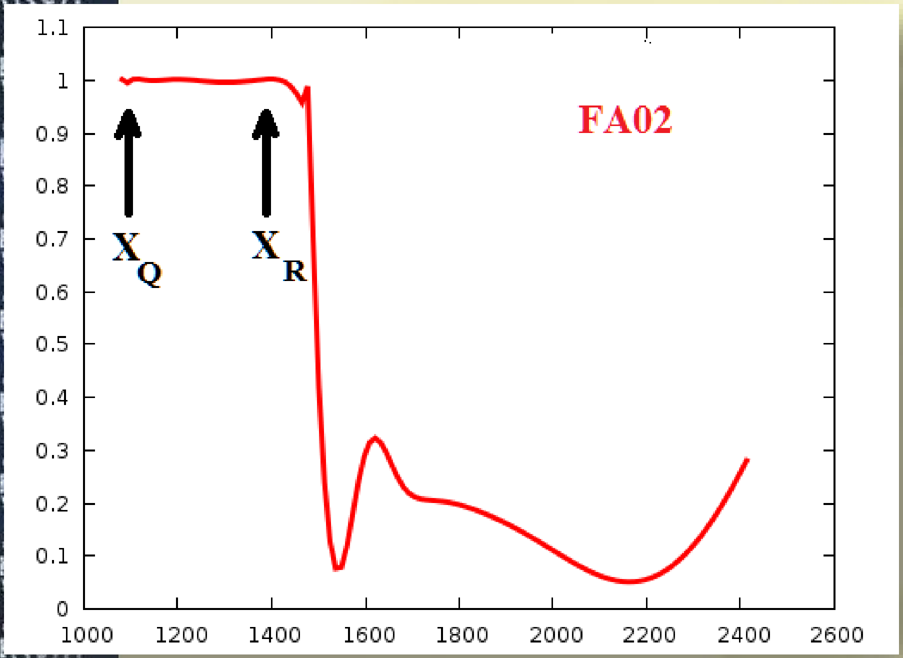


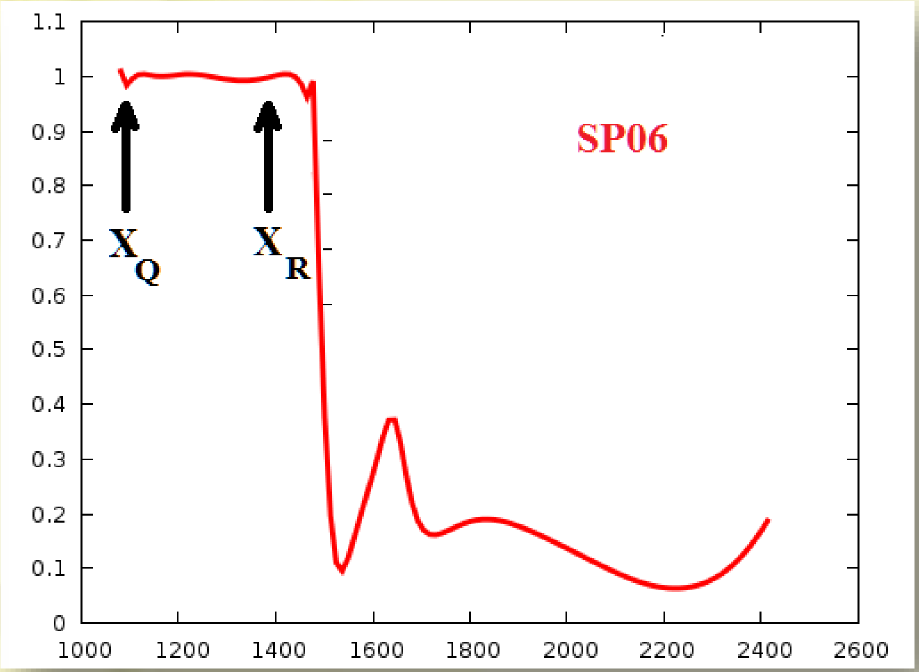
2. Unitarity

Elestic unitarity is introduced via penalty function

$$\chi^2 = \sum_{j=1}^{N_{pts}} | T^{inp}(\omega_j) - T(\omega_j) |^2 / \mathrm{w}_j^2 + \sum_{j=1}^{3} \lambda^j \chi^j_{Pen} + \beta \sum_{\mathbf{j=1}}^{\mathbf{N^{el}_{pts}}} (\mathbf{1} - \mathbf{S}(\omega_\mathbf{j}))\mathbf{S}(\omega_\mathbf{j})^\dagger)$$

Unitarity test





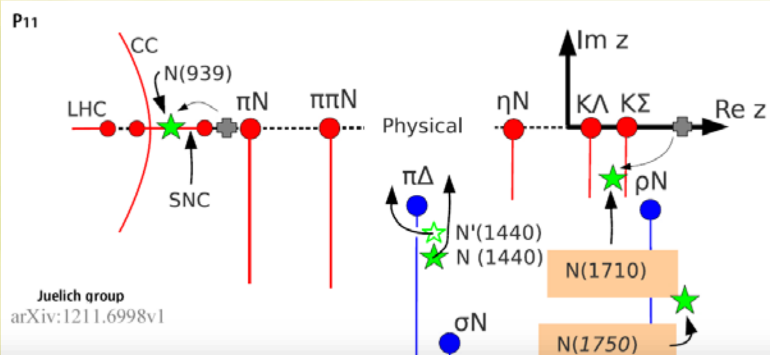
The model

We use Mittag-Leffleur decomposition of „analyzed" function:

$$T(\omega) = \sum_{i=1}^{k} \frac{a_{-1}^{(i)}}{\omega_i - \omega} + B^L(\omega)$$

regular background

k - simple poles



We know analytic properties (number and position of cuts) of analyzed fu

$$B^L(\omega) = \sum_{n=0}^{M} c_n \, Z(\omega)^n + \sum_{n=0}^{N} d_n \, W(\omega)^n + \cdots$$

$$Z(\omega) = \frac{\alpha - \sqrt{x_P - \omega}}{\alpha + \sqrt{x_P - \omega}}; \quad W(\omega) = \frac{\beta - \sqrt{x_Q - \omega}}{\beta + \sqrt{x_Q - \omega}} + \cdots$$

$$a_{-1}^{(i)}, \omega_i, \omega \in \mathbb{C}$$

$$c_n, x_P, d_n, x_Q, \alpha, \beta ... \in \mathbb{R}$$

$$\text{and } k, M, N ... \in \mathbb{N}. \tag{4}$$

ONE Pietarinen power series per cut

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Method has problems, and the one of them definitely is:

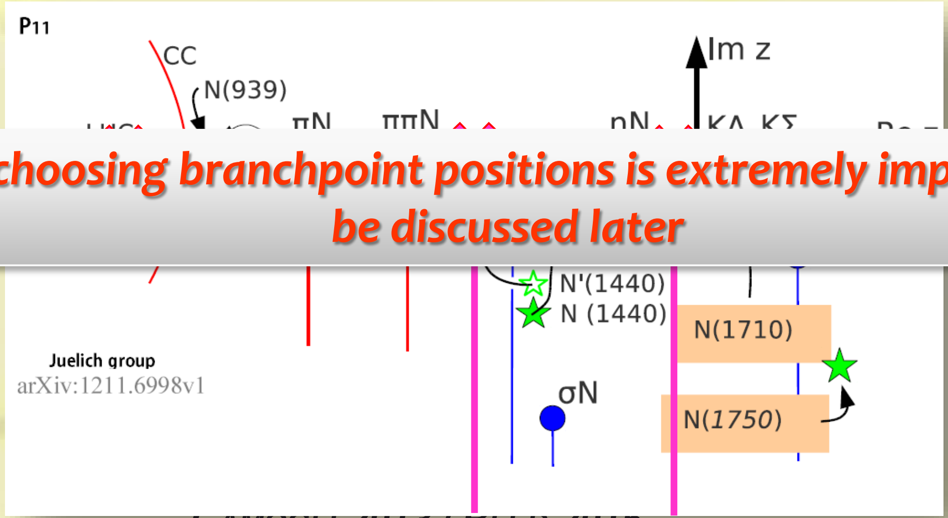
There is a lot of cuts, so it is difficult to imagine that we shall be able to represent each cut with one Pietarinen series (too many possibly interfering terms).

Answer:

We shall use „effective” cuts to represent dominant effects.

We use three Pietarinen series:

- ***One to represent subthreshold, unphysical contributions***
- ***Two in physical region to represent all inelastic channel openings***



Strategy of choosing branchpoint positions is extremely important and will be discussed later

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Advantage:

The method is „self-checking" !

It might not work.
But, if it works, and if we obtain a good χ^2, then we have obtained

AN ANALYTIC FUNCTION WITH WELL KNOWN POLES AND CUTS WHICH DEFINITELY DESCRIBES THE INPUT!

So, if we have disagreements with other methods, then we are looking at two different analytic functions which are almost identical on a discrete set, so we may discuss the general stability of the problem.

However, our solution definitely IS A SOLUTION!

What can we do with this model?

1. <u>We may analyze various kinds of inputs</u>

   a. Theoretical curves coming from ANY model

   but also

   b. Information coming directly from experiment (partial wave data)

Observe: Partial wave data are much more convenient to analyze!

To fit „theoretical input" we have to „guess" both:
pole position AND exact analyticity structure of the background imposed by the analyzed model

To fit „experimental input" we have to „guess" only:
pole position AND the simplest analyticity structure of the background as no information about functional type is imposed

Does it work?

Testing is a very simple procedure. It comes to:

Doesn't work
Works

TESTING

a. *Testing on a toy model:* *arXiv nucl-th 1212.1295*

b. *Testing and application on realistic amplitudes*

   i. *πN elastic scattering*
      a. *ED PW amplitudes (some solutions from GWU/SAID)*
      b. *ED PW amplitudes (some solutions from Dubna-Mainz-Taipei)*
   ii. *Photo – and electroproduction on nucleon*
      a. *ED multipoles (all solutions from MAID and SAID)*
      b. *SES multipoles (all solutions from MAID and SAID)*

a. Toy model

We have constructed a toy model using two poles and two cuts, used it to construct the input data set, attributed error bars of 5%, and tried to use L+P method to extract pole parameters under different conditions.

$$T^{ty}(\omega) = \sum_{i=1}^{2} \frac{r_i^{ty} + \imath\, g_i^{ty}}{M_i^{ty} - \omega - \imath\, W_i^{ty}} + \qquad (5)$$

$$+ \; C_1\,\Phi(\omega, 0.25) + C_2\,\Phi(\omega, 1.) + B^{ty}(\omega),$$

$$\Phi(\omega, a) = \frac{\sqrt{\omega(-4a+\omega)}}{2\pi\omega} \ln \frac{2a - \omega - \sqrt{\omega(-4a+\omega)}}{2a}$$

$$B^{ty}(\omega) = B_1 \frac{10.}{-10. - \omega - \imath\, 5.} + B_2 \frac{10.}{-6. - \omega - \imath\, 4.},$$

where

$$r_i^{ty}, g_i^{ty}, M_i^{ty}, W_i^{ty} \in \mathbb{R}.$$

$$\Phi(x, a) = \frac{x - x_0}{\pi} \int_{x_0}^{\infty} \frac{\Re e(x', a)}{(x' - x)(x' - x_0)}\, dx'$$

where

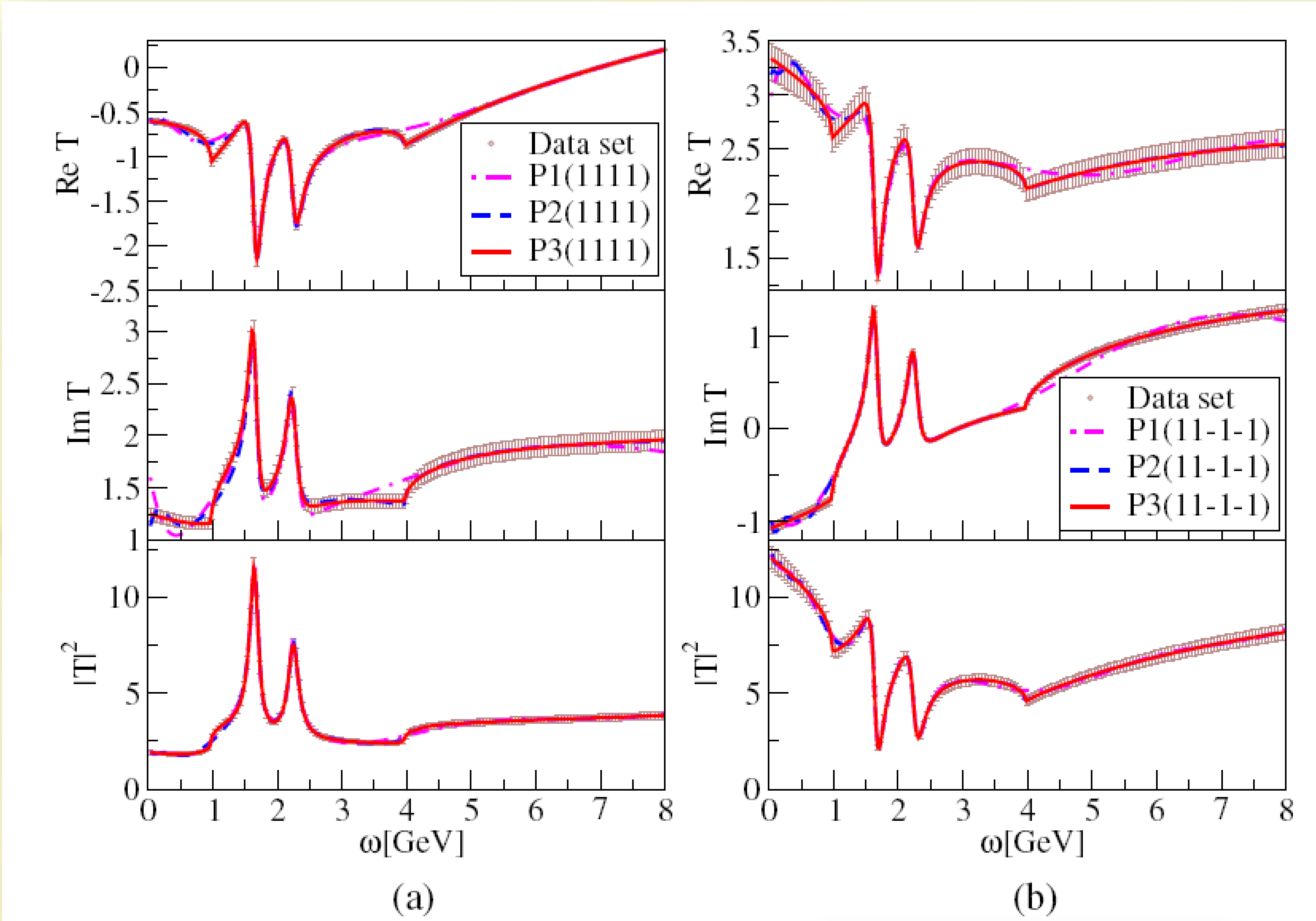
$$\Re e(x, a) = \sqrt{x^2 - 4ax}/2x$$

cut is at $x_0 = 4a$

$$\Gamma_i = -2\, W_i.$$

C_1, C_2, B_1, B_2 = -1, 0, 1

r_1	g_1	M_1	Γ_1	r_2	g_2	M_2	Γ_2
0.1	0.09	1.65	0.165	0.09	0.06	2.25	0.2



$B_1 = 1,\ B_2 = 1$

$B_1 = \text{-}1,\ B_2 = \text{-}1$

C_1	C_2	B_1	B_2	r_1	g_1	M_1	Γ_1	r_2	g_2	M_2	Γ_2	α	x_P	N_1	β	x_Q	N_2	γ	x_R	N_3	$10^2\chi^2_R$
Toy-model																					
				0.1	**0.09**	**1.65**	**0.165**	**0.09**	**0.06**	**2.25**	**0.2**										
Fitted results																					
Strategy a.																					
1	0	0	0	0.100	0.089	1.649	0.165	0.090	0.060	2.249	0.200	2.48	0.97	5							0.03
0	1	0	0	0.099	0.090	1.650	0.165	0.090	0.060	2.249	0.199	3.97	3.97	5							0.01
0	0	1	1	0.098	0.091	1.650	0.165	0.090	0.060	2.250	0.200	1.19	-14.94	7							0.2
0	0	-1	-1	0.099	0.089	1.649	0.1649	0.089	0.059	2.249	0.199	0.99	-9.63	7							0.01
1	0	1	1	0.103	0.100	1.653	0.171	0.101	0.067	2.249	0.221	0.71	-0.23	11							28
1	0	1	1	0.099	0.090	1.650	0.164	0.089	0.060	2.250	0.199	-2.04	-17.58	5	4.27	0.97	5				0.28
1	0	- 1	- 1	0.097	0.087	1.651	0.161	0.090	0.060	2.250	0.201	0.90	-0.39	20							22.0
1	0	-1	-1	0.099	0.089	1.649	0.164	0.090	0.059	2.249	0.199	2.96	-8.97	6	1.56	0.97	6				1.00
0	1	1	1	0.107	0.088	1.646	0.166	0.093	0.048	2.239	0.197	2.06	-0.89	10							114.79
0	1	1	1	0.099	0.090	1.650	0.165	0.090	0.060	2.250	0.200	1.94	-16.33	5	6.42	3.97	5				0.02
0	1	-1	-1	0.090	0.086	1.651	0.156	0.095	0.058	2.248	0.202	0.969	-0.37	12							238.38
0	1	-1	-1	0.099	0.090	1.650	0.165	0.090	0.060	2.250	0.200	0.81	-7.89	8	1.24	3.97	8				0.06
1	1	1	1	0.085	0.102	1.663	0.171	0.087	0.075	2.262	0.216	1.09	-2.64	10							328.19
1	1	1	1	0.098	0.086	1.650	0.161	0.095	0.058	2.247	0.199	0.44	-0.47	9	1.95	3.97	8				70.37
1	1	1	1	0.099	0.090	1.650	0.164	0.089	0.061	2.251	0.200	4.19	-22.99	5	2.22	3.98	5	1.67	0.97	3	0.24
1	1	-1	-1	0.090	0.105	1.657	0.182	0.078	0.061	2.260	0.189	1.38	-3.12	10							467.54
1	1	-1	-1	0.095	0.098	1.654	0.173	0.086	0.061	2.254	0.198	0.61	-0.20	9	25.91	3.98	8				60.94
1	1	-1	-1	0.100	0.090	1.650	0.165	0.090	0.060	2.250	0.200	1.85	-6.25	3	16.36	3.97	3	1.32	0.98	3	0.72

b. Testing on realistic amplitude

- ***πN elastic***
  - ***GWU/SAID FA02***
  - ***GWU/SAID SP06***
  - ***GWU/SAID WI08***
  - ***DMT***

- ***Photoproduction***
  - ***GWU/SAID ZN11 ED***

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Quality of the fit

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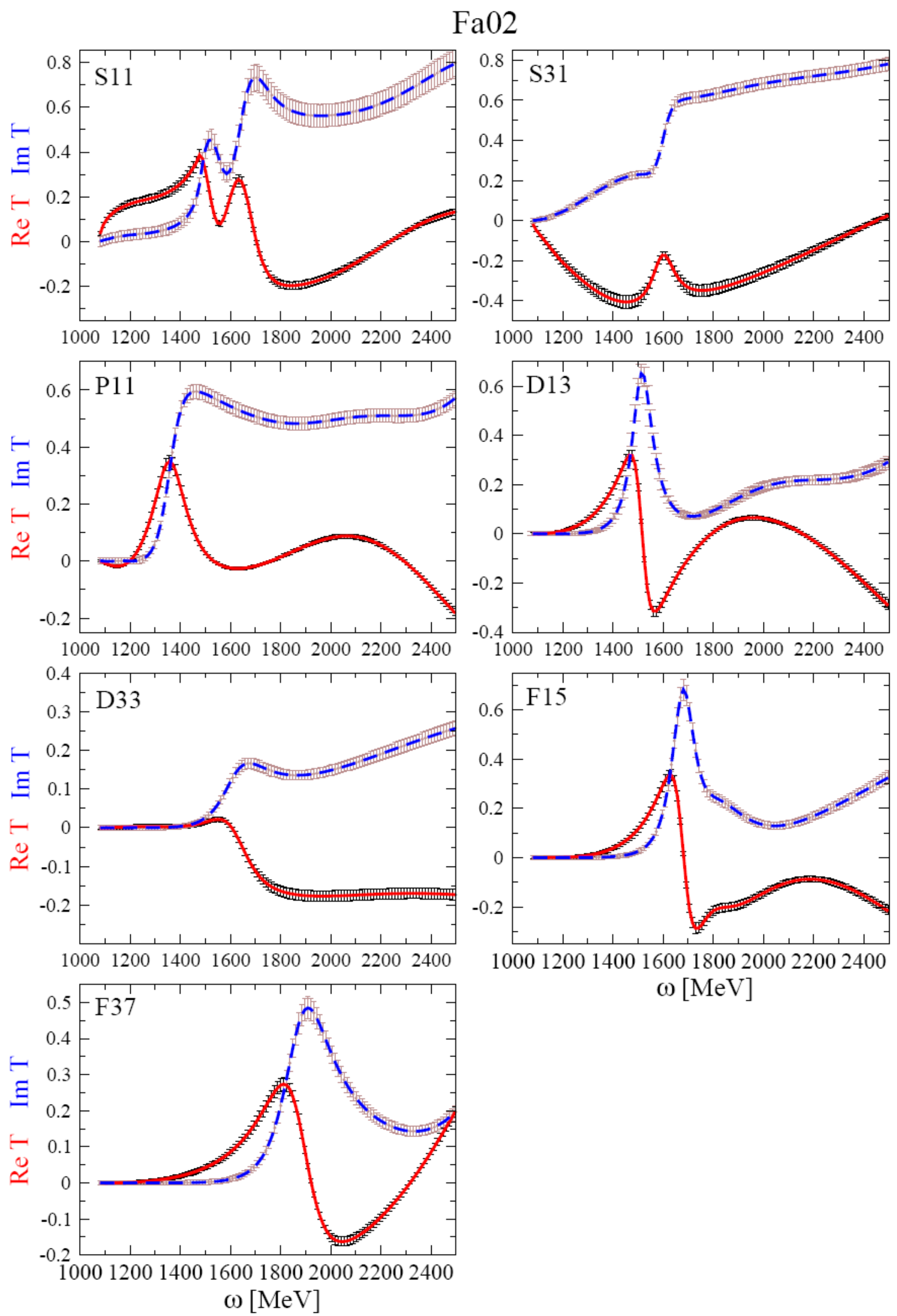
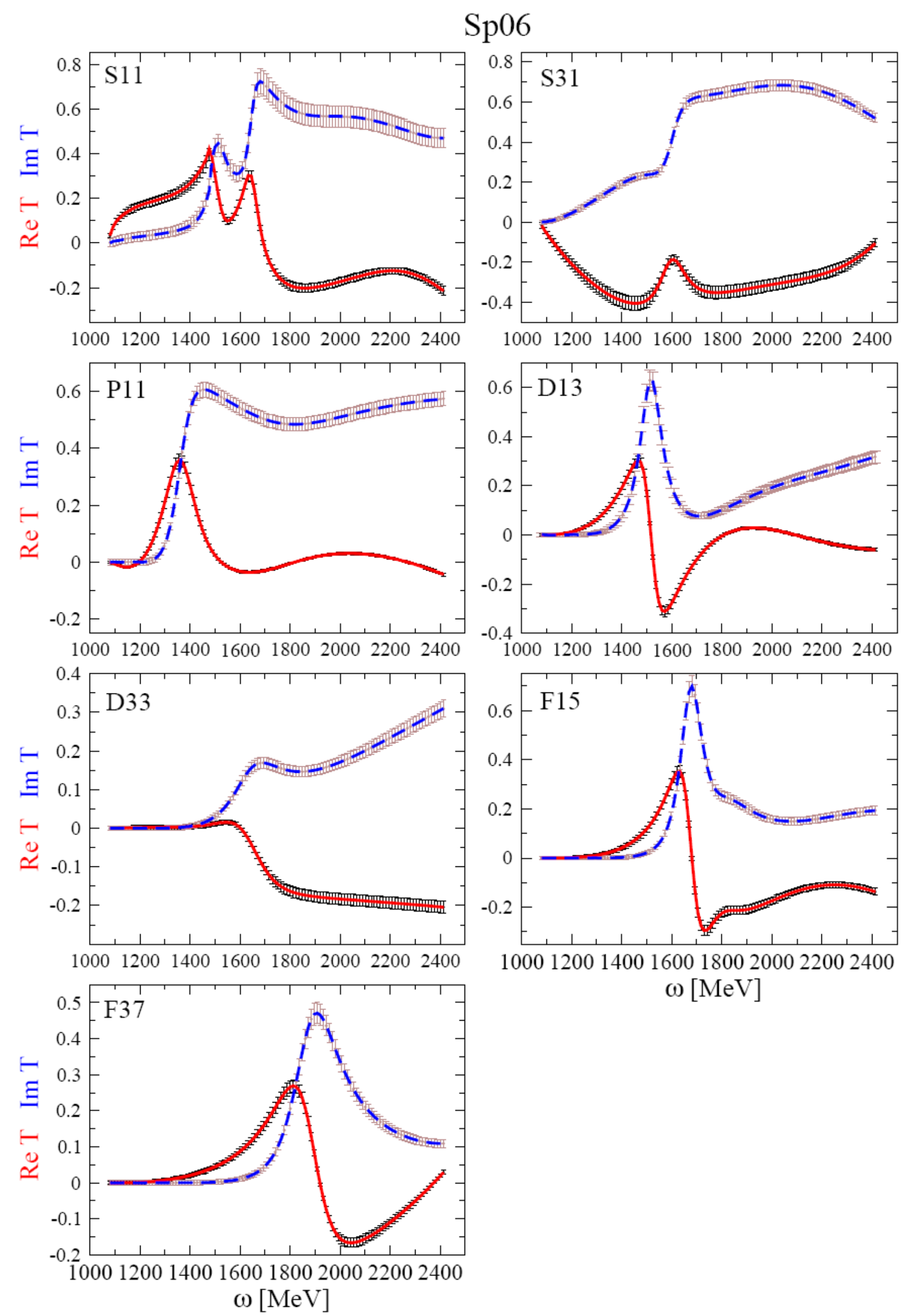
πN elastic scattering
SAID FA02 ED



CAM

πN elastic scattering
SAID SP06 ED



CAMO

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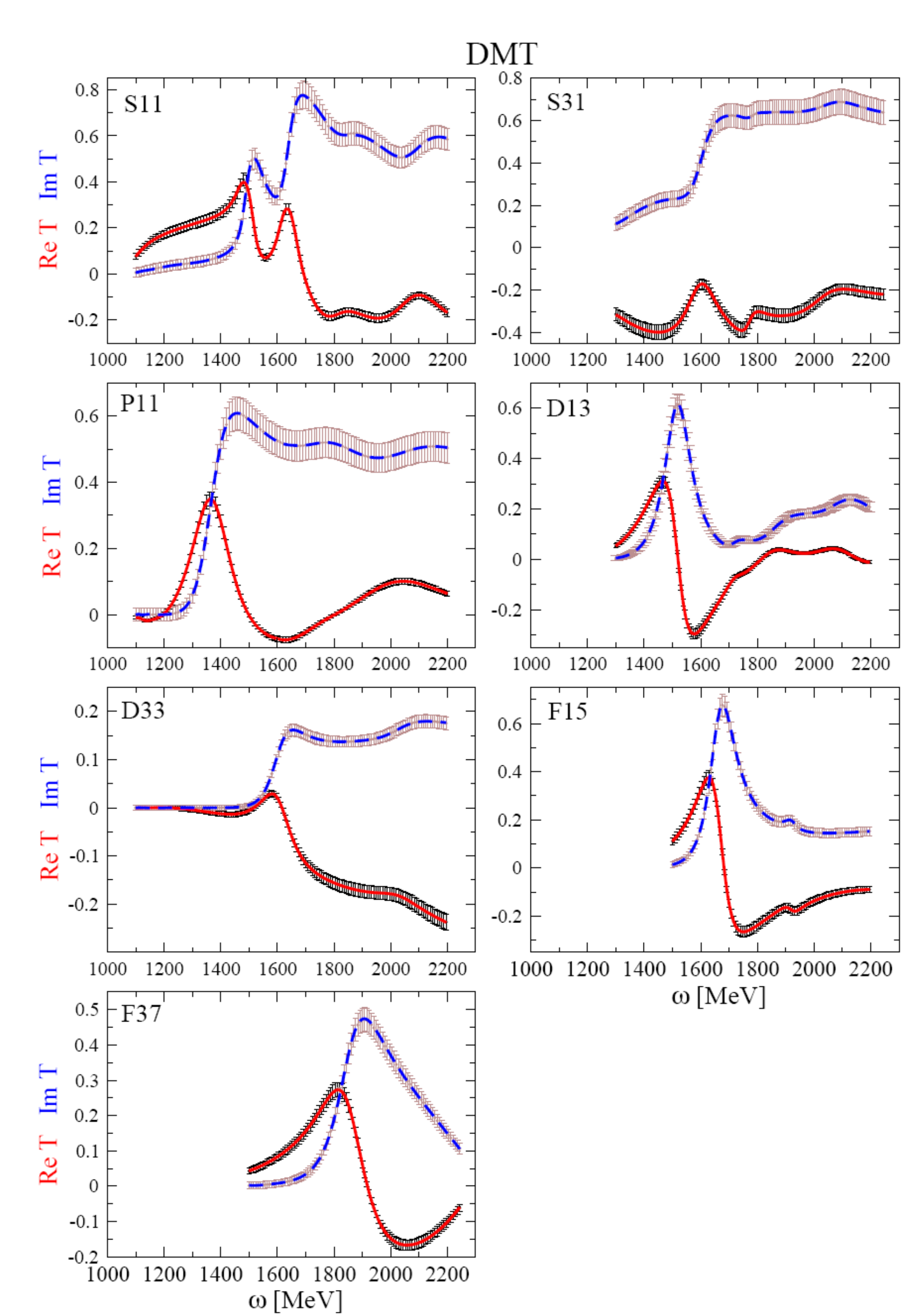
πN elastic scattering DMT



CAMOC

PW	Solution	M_1	Γ_1	$\|a_1\|$	θ_1^0	M_{2-4}	Γ_{2-4}	$\|a_{2-4}\|$	θ_{2-4}^0	$x_P/10^3$	$x_Q\ /10^3$	$x_R/10^3$	$10^2\chi_R^2$
S_{11}	DMT [3, 4]	**1499**	**78**	**14**	**-45**	**1631**	**120**	**35**	**-83**				
						1733	**180**	**16**	**-29**				
	DMT [22]					**2027**	**180**	**23**	**-150**				
	DMT L+P	1500	76	13.4	-46	1636	99	22	-94				
						1810	164	9.6	-176				
						2077	220	22.5	-122	1.0	1.077	1.486	0.6
S_{31}	DMT [3, 4]	**1598**	**148**	**23**	**-98**	**1774**	**72**	**3.8**	**-181**				
						1984	**254**	**26**	**-170**				
	DMT L+P	1597	140	21	-104	1771	69	2.2	-172				
						2040	195	7	-109	-11.476	1.077	1.739	0.2
P_{11}	DMT [3, 4]	**1371**	**190**	**50**	**-79**	**1746**	**368**	**11**	**-54**				
	DMT [22]					**1997**	**458**	**56**	**-145**				
	DMT L+P	1370	190	50	-81	1763	235	5	-56				
						2015	467	36	-99	0.699	1.077	1.537	0.05
D_{13}	DMT [3, 4]	**1515**	**120**	**40**	**-7**	**1718**	**96**	**2.8**	**-91**				
						1854	**214**	**16**	**-96**				
	DMT [22]					**2099**	**216**	**13**	**-58**				
	DMT L+P	1517	120	40	-5	1721	89	2.1	-76				
						1858	228	15	-87				
						2101	231	14	-49	1.00	1.077	1.266	0.32
D_{33}	DMT [3, 4, 22]	**1604**	**142**	**9.4**	**-63**	**2042**	**254**	**4.84**	**-75**				
	DMT L+P	1605	141	9.3	-63	2023	241	4	-93	0.623	1.077	1.324	0.06
F_{15}	DMT [3, 4]	**1664**	**114**	**38**	**-26**	**1919**	**52**	**1.0**	**15**				
	DMT L+P	1664	114	38	-26	1920	52	1.0	16	0.7	1.077	1.225	0.02
F_{37}	DMT [3, 4]	**1858**	**208**	**43**	**-48**								
	DMT L+P	1858	207	43	-49					-3.999	1.077	1.223	0.48

PW	Solution	M_1	Γ_1	$\|a_1\|$	θ_1^0	M_{2-4}	Γ_{2-4}	$\|a_{2-4}\|$	θ_{2-4}^0	$x_P/10^3$	$x_Q/10^3$	$x_R/10^3$	$10^2\chi_R^2$
S_{11}	FA02 [1]	**1526**	**130**	**33**	**14**	**1653**	**182**	**69**	**-55**				
	FA02 L+P	1518	121	17	-32	1656	182	68	-39	-60.1	1.077	1.471	0.32
	SP06 [1]	**1502**	**95**	**16**	**-16**	**1648**	**80**	**14**	**-69**				
	SP06 L+P	1509	96	15	-21	1645	80	14	-80	-29.5	1.077	1.479	2.90
	WI08 [2]	**1499**	**98**	**-**	**-**	**1647**	**84**	**-**	**-**				
						1666	**520**	**-**	**-**				
	WI08 L+P	1504	78	11	-60	1644	86	17	-83				
						1669	517	419	-74	-0.339	1.077	1.483	3.57
S_{31}	FA02 [1]	**1594**	**118**	**17**	**-104**								
	FA02 L+P	1596	112	15	-101					-59.3	1.077	1.183	0.48
	SP06 [1]	**1595**	**135**	**15**	**-92**								
	SP06 L+P	1596	133	18	-105					-16.7	1.077	1.309	0.35
	WI08 [2]	**1594**	**136**	**-**	**-**								
	WI08 L+P	1598	130	18	-104					-92.7	1.077	1.589	0.57
P_{11}	FA02 [1]	**1357**	**160**	**36**	**-102**								
	FA02 L+P	1354	169	38	-98					-100	1.077	1.202	0.66
	SP06 [1]	**1359**	**162**	**38**	**-98**								
	SP06 L+P	1358	183	53	-92					-62.1	1.077	1.215	0.09
	WI08 [2]	**1358**	**160**	**-**	**-**								
	WI08 L+P	1357	177	47	-93					-98.9	1.077	1.202	0.07
D_{13}	FA02 [1]	**1514**	**102**	**35**	**-6**								
	FA02 L+P	1513	101	34	-9					-67.4	1.077	1.222	0.85
	SP06 [1]	**1515**	**113**	**38**	**-5**								
	SP06 L+P	1515	113	38	-6					-50.1	1.077	1.216	0.57
	WI08 [2]	**1515**	**110**	**-**	**-**								
	WI08 L+P	1515	111	38	-5					-81.1	1.077	1.169	0.15
D_{33}	FA02 [1]	**1617**	**226**	**16**	**-47**								
	FA02 L+P	1618	227	16	-47					-27.3	1.077	1.204	0.008
	SP06 [1]	**1632**	**253**	**18**	**-48**								
	SP06 L+P	1635	251	18	-37					-54.1	1.077	1.198	0.009
	WI08 [2]	no results											
	WI08 L+P												
F_{15}	FA02 [1]	**1678**	**120**	**43**	**1**	**1779**	**248**	**47**	**-61**				
	FA02 L+P	1679	118	42	-5	1779	245	31	-84	1.032	1.077	1.549	0.64
	SP06 [1]	**1674**	**115**	**42**	**-4**	**1785**[a]	**244**[b]	**60**	**-67**				
	SP06 L+P	1673	116	43	-14	1776	226	24	-98	-8.99	1.077	1.301	0.08
	WI08 [2]	**1674**	**114**	**-**	**-**	**1779**	**276**	**-**	**-**				
	WI08 L+P	1675	115	44	-8	1776	233	34	-99	-51.7	1.077	1.726	0.28
F_{37}	FA02 [1]	**1874**	**236**	**57**	**-34**								
	FA02 L+P	1874	236	55	-35					-14.9	1.077	1.739	0.04
	SP06 [1]	**1876**	**227**	**53**	**-31**								
	SP06 L+P	1876	226	53	-31					-32.8	1.077	1.137	0.06
	WI08 [2]	**1883**	**230**	**-**	**-**								
	WI08 L+P	1874	227	55	-35					-37.8	1.077	1.736	0.03

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$N(1860)$ POLE POSITION

REAL PART

VALUE (MeV)	DOCUMENT ID		TECN	COMMENT
1830^{+120}_{-60}	ANISOVICH	12A	DPWA	Multichannel
~~1807~~ 1785	ARNDT	06	DPWA	$\pi N \to \pi N, \eta N$

$-2\times$IMAGINARY PART

VALUE (MeV)	DOCUMENT ID		TECN	COMMENT
250^{+150}_{-50}	ANISOVICH	12A	DPWA	Multichannel
~~109~~ 244	ARNDT	06	DPWA	$\pi N \to \pi N, \eta N$

$N(1860)$ ELASTIC POLE RESIDUE

MODULUS $|r|$

VALUE (MeV)	DOCUMENT ID		TECN	COMMENT
50 ± 20	ANISOVICH	12A	DPWA	Multichannel
~~60~~ 43	ARNDT	06	DPWA	$\pi N \to \pi N, \eta N$

PHASE θ

VALUE (°)	DOCUMENT ID		TECN	COMMENT
-80 ± 40	ANISOVICH	12A	DPWA	Multichannel
~~-67~~ -64	ARNDT	06	DPWA	$\pi N \to \pi N, \eta N$

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Photoproduction

GWU/SAID Zn11 ED solution

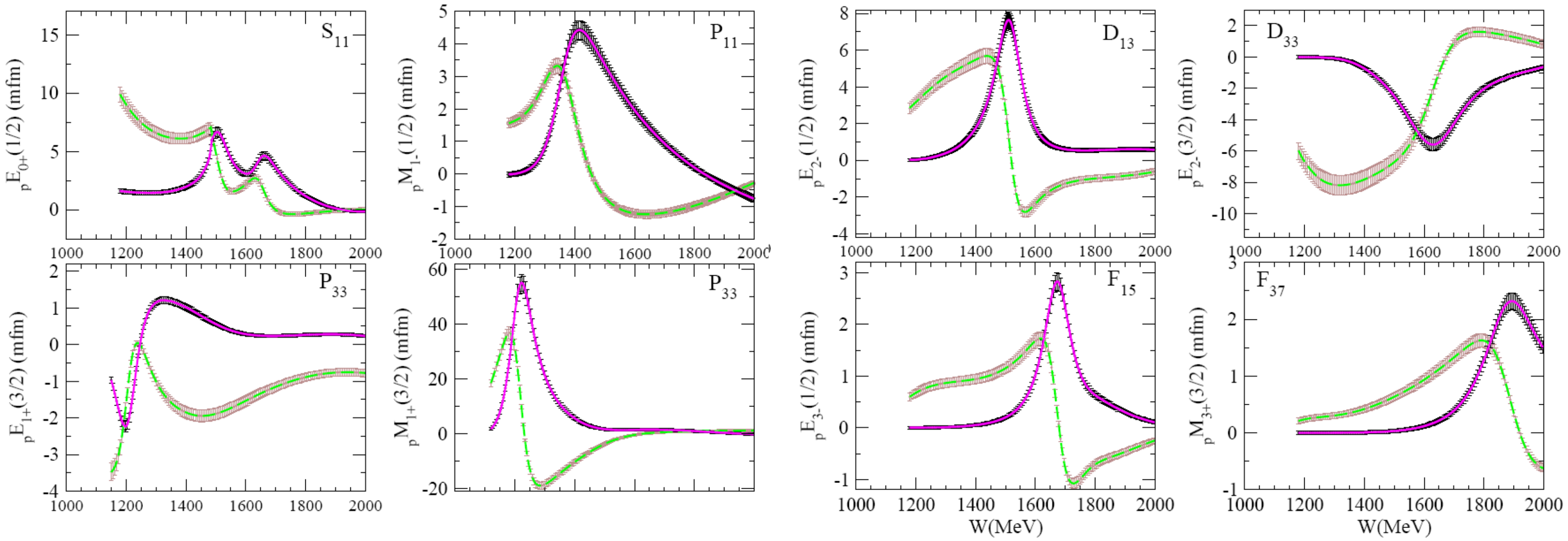

FIG. 1: L+P fit to GWU/SAID ZN11 ED solutions.

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TABLE I: Pole positions in MeV and residues of multipoles as moduli in mfm · GeV and phases in degrees. The results from L+P expansion are given for GW/SAID ZN11 energy dependent (ED), and ZN11 *a.c.* shows the result of analytical continuation into the complex region.

Multipole	Source	Resonance	$\mathrm{Re}\, W_p$	$-2\mathrm{Im}\, W_p$	|residue|	θ	Resonance	$\mathrm{Re}\, W_p$	$-2\mathrm{Im}\, W_p$	|residue|	θ
$S_{11}({}_pE_{0+})$	ZN11 ED	$N(1535)\ 1/2^-$	1506	90	0.203	$-7°$	$N(1650)\ 1/2^-$	1648	88	0.128	$-51°$
	ZN11 *a.c.*		1502	94	0.25	$-3°$		1648	80	0.100	$-50°$
$P_{11}({}_pM_{1-})$	ZN11 ED	$N(1440)\ 1/2^+$	1363	173	0.353	$-68°$					
	ZN11 *a.c.*		1362	164	0.34	$-84°$					
$P_{33}({}_pE_{1+})$	ZN11 ED	$\Delta(1232)\ 3/2^+$	1210	98	0.182	$-150°$	$\Delta(1600)\ 3/2^+$	1494	444	0.281	$139°$
	ZN11 *a.c.*		1210	98	0.18	$-149°$		1457	400	0.146	$80°$
$P_{33}({}_pM_{1+})$	ZN11 ED	$\Delta(1232)\ 3/2^+$	1211	99	2.969	$-26°$					
	ZN11 *a.c.*		1210	98	2.98	$-28°$					
$D_{13}({}_pE_{2-})$	ZN11 ED	$N(1520)\ 3/2^-$	1516	112	0.427	$12°$					
	ZN11 *a.c.*		1515	112	0.440	$11°$					
$D_{33}({}_pE_{2-})$	ZN11 ED	$\Delta(1700)\ 3/2^-$	1634	253	0.721	$-171°$					
	ZN11 *a.c.*		1632	252	0.719	$-174°$					
$F_{15}({}_pE_{3-})$	ZN11 ED	$N(1680)\ 5/2^+$	1673	115	0.169	$1°$	$N(2000)\ 5/2^+$	1792	202	0.033	$-53°$
	ZN11 *a.c.*		1673	114	0.169	$8°$		~~1807~~ **1785**	~~199~~ **244**	0.126	$-58°$
$F_{37}({}_pM_{3+})$	ZN11 ED	$\Delta(1950)\ 7/2^+$	1876	227	0.255	$-16°$					
	ZN11 *a.c.*		1876	226	?	?°					

Error analysis

The only problem in the model are thresholds. Their number is definitely at this moment insufficient, so we must propose a stretegy.

Namely, if we fail to reproduce background exactly (and that we certainly do as soon as number of thresholds is insufficient), the pole terms try to compensate for the approximation made.

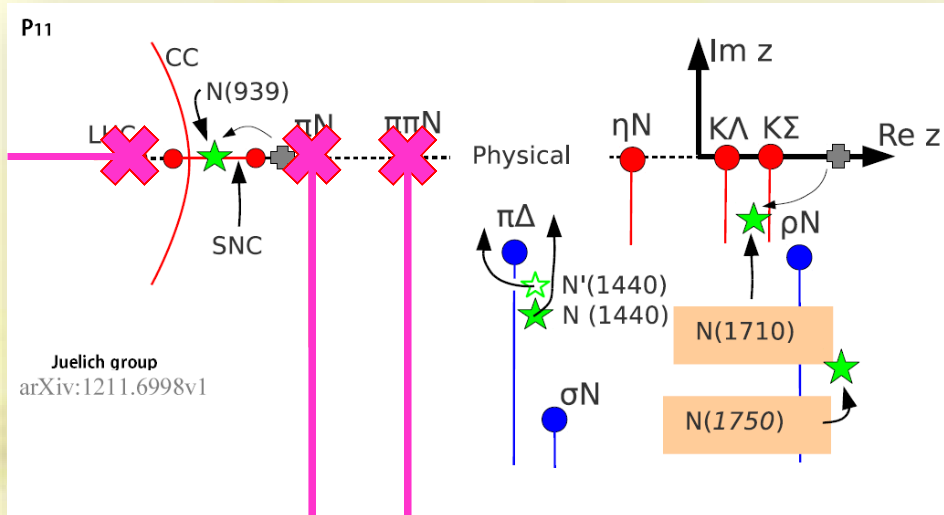
$$T(\omega) = \sum_{i=1}^{k} \frac{a_{-1}^{(i)}}{\omega_i - \omega} + B^L(\omega)$$

We propose two strategies:

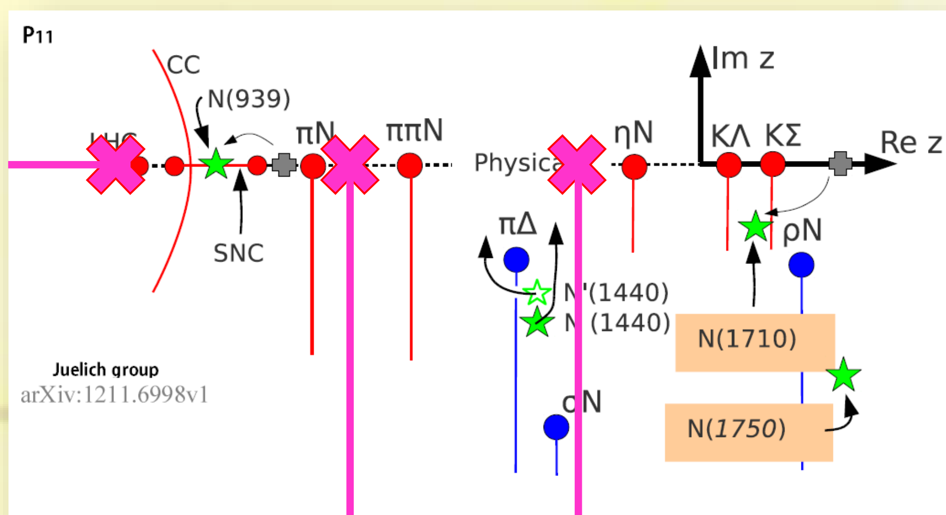
1. ***To fix the pole at the values expected to dominate for a chosen channel***
2. ***To allow poles to vary as a fitting parameter and allow the fit to find optimal choice of two effective thresholds which will replace the exact values***

In practice this looks like that:

Option 1:



Option 2:



Example of the error estimate:

$P_{33}(M_{1+})$ Threshold positions

Red color - Results obtained with variable thresholds

Blue color - Results obtained with fixed physical thresholds

$P_{33}(M_{1+}(\frac{3}{2}))$		Re	$-2Im$	(Re, Im) $(\|r_1\|, \theta_1)$ [mfm]	χ^2_R	x_P	x_Q	x_R
sn11	Sol1	1210	102	$(2.737, -1.505)$ $(3.123, -29)$	0.0014	86	1151	1319
	Sol2	1211	99	$(2.633, -1.337)$ $(2.953, -27)$	0.006	76	1166	1294
	Sol3	1212	98	$(2.600, -1.208)$ $(2.867, -25)$	0.008	-4571	1077	1215
	Sol4	1211	102	$(2.760, -1.432)$ $(3.109, -27)$	0.006	-489	1077	1371
Our estimate		1210.51 ± 0.39	101.15 ± 0.86					

$(2.711 \pm 0.03, -1.439 \pm 0.055)$
$(3.069 \pm 0.071, -28. \pm 0.18)$

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i}; \; w_i = \frac{\chi^2_{min}}{\chi^2_i} \; ; \; w_i = \frac{1}{\sigma_i^2} \; ; \; \sigma_{\bar{x}}^2 = \frac{1}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}} \frac{1}{(n-1)} \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma_i^2}$$

We used weighted average.

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Conclusion

The L+P method defined as:

$$
\begin{aligned}
T(\omega) &= \sum_{i=1}^{k} \frac{a_{-1}^{(i)}}{\omega_i - \omega} + B^L(\omega) \\
B^L(\omega) &= \sum_{n=0}^{M} c_n\, Z(\omega)^n + \sum_{n=0}^{N} d_n\, W(\omega)^n + \cdots \\
Z(\omega) &= \frac{\alpha - \sqrt{x_P - \omega}}{\alpha + \sqrt{x_P - \omega}}; \quad W(\omega) = \frac{\beta - \sqrt{x_Q - \omega}}{\beta + \sqrt{x_Q - \omega}} + \cdots \\
& a_{-1}^{(i)}, \omega_i, \omega \in \mathbb{C} \\
& c_n, x_P, d_n, x_Q, \alpha, \beta ... \in \mathbb{R} \\
& \text{and } \; k, M, N ... \in \mathbb{N}. \qquad (4)
\end{aligned}
$$

WORKS

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World recognition

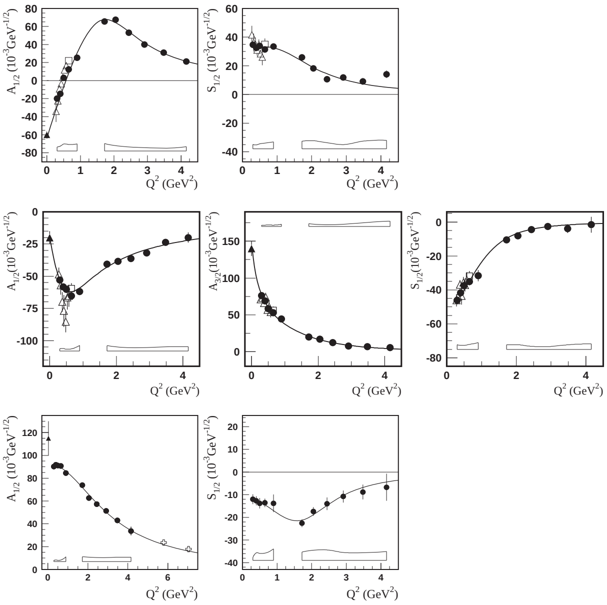

Figure 2: Transverse and scalar (longitudinal) helicity amplitudes for $\gamma p \to N(1440)1/2^+$ (top), $\gamma p \to N(1520)3/2^-$ (center), and $\gamma p \to N(1535)1/2^-$ (bottom) as extracted from the JLab/CLAS data in $n\pi^+$ production (full circles), in $p\pi^+\pi^-$ (open triangles), combined single and double pion production (open squares). The solid triangle is the PDG 2013 value at $Q^2 = 0$. The open boxes are the model uncertainties of the full circles. The figures are kindly provided by V. Burkert, JLab.

$A_{1/2}$ is small at the photon point, increases rapidly with Q^2 and then falls off with $\sim Q^{-3}$. Quantitative agreement with the data is, however, achieved only when meson cloud effects are included.

At high Q^2, both amplitudes for $N(1440)1/2^+$ are qualitatively described by light front quark models [22]: at short distances the resonance behaves as expected from a radial excitation of the nucleon. On the other hand, $A_{1/2}$ changes sign at about 0.6 GeV^2. This remarkable behavior has not been observed before for any nucleon form factor or transition amplitude. Obviously, an important change in the structure occurs when the resonance is probed as a function of Q^2.

The Q^2 dependence of $A_{1/2}$ of the $N(1535)1/2^-$ resonance exhibits the expected $\sim Q^{-3}$ dependence, except for small Q^2 values where meson cloud effects set in.

VII. Partial wave analyses

Several PWA groups are now actively involved in the analysis of the new data. The GWU group maintains a nearly complete database covering reactions from πN and KN elastic scattering to $\gamma N \to N\pi$, $N\eta$, and $N\eta'$. It is presently the only group determining πN elastic amplitudes from scattering data in sliced energy bins. Given the high-precision of photoproduction data already or soon to be collected, the spectrum of N and Δ resonances will in the near future be better known.

Fits to the data are performed by various groups with the aim to understand the reaction dynamics and to identify N and Δ resonances. For practical reasons, approximations have to be made. We mention several analyses here: (1) The Mainz unitary isobar model [23] focusses on the correct treatment of the low-energy domain. Resonances are added to the unitary amplitude as a sum of Breit-Wigner amplitudes. This model also obtains resonance transition form factors and helicity amplitudes from electroproduction [19]. (2) For $N\pi$ electroproduction, the Yerevan/JLab group uses both the unitary isobar model and the dispersion relation approach developed in [22]. A phenomenological model was developed to extract resonance couplings and partial decay widths from exclusive $\pi^+\pi^- p$ electroproduction [21]. (3) Multichannel analyses using K-matrix parameterizations derive background terms from a chiral Lagrangian - providing a microscopical description of the background - (Giessen [24,25]) or from phenomenology (Bonn-Gatchina [26]) . (4.) Several groups (EBAC-Jlab [27,28], ANL-Osaka [29], Dubna-Mainz-Taipeh [30], Bonn-Jülich [31,32,33], Valencia [34]) use dynamical reaction models, driven by chiral Lagrangians, which take dispersive parts of intermediate states into account. Several other groups have made important contributions. The Giessen group pioneered multichannel analyses of large data sets on pion- and photo-induced reactions [24,25]. The Bonn-Gatchina group included recent high-statistics data and reported systematic searches for new baryon resonances in all relevant partial waves. A summary of their results can be found in Ref. [26].

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