The Factorization Method for the Reconstruction of Inclusions

Martin Hanke

Institut für Mathematik
Johannes Gutenberg-Universität Mainz

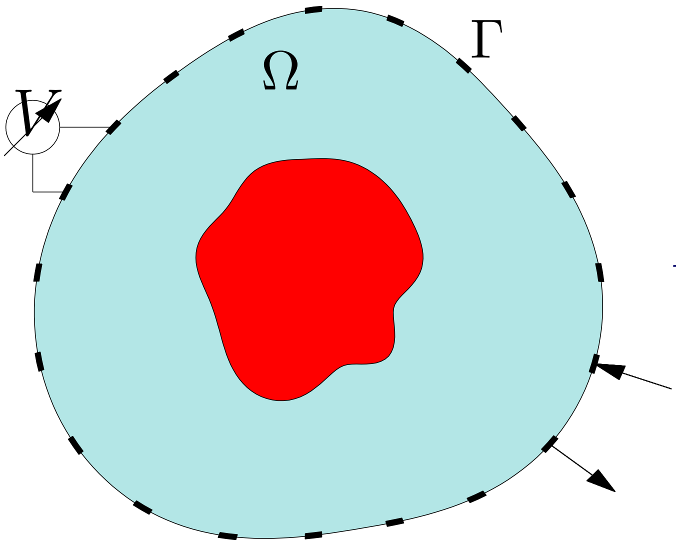
hanke@math.uni-mainz.de

January 2007

Overview

- Electrical Impedance Tomography
- Factorization Method
- Applications
- Implementation

Impedance Tomography



σ	:	electric conductivity
u	:	electric potential
$E = -\operatorname{grad} u$	:	electric field
$J = \sigma E$	:	current field (Ohm's law)
f	:	imposed boundary current

$\rightsquigarrow$

$$\begin{aligned} \operatorname{div}(\sigma \operatorname{grad} u) &= 0 && \text{in } \Omega \\ \sigma \frac{\partial u}{\partial \nu} &= f && \text{on } \Gamma \end{aligned}$$

Neumann-Dirichlet-Operator

$\{f_j\}$: current pattern (basis of $\mathcal{L}^2_\diamond(\Gamma)$) $\qquad \int_\Gamma f_j(\theta)\,d\theta = 0$

$\{g_j\}$: boundary potential on Γ $\qquad \int_\Gamma g_j(\theta)\,d\theta = 0$

Neumann-Dirichlet-Operator

$$\Lambda(\sigma) : \begin{cases} \mathcal{L}^2_\diamond(\Gamma) & \longrightarrow & \mathcal{L}^2_\diamond(\Gamma) \\ f_j & \longmapsto & g_j \end{cases}$$

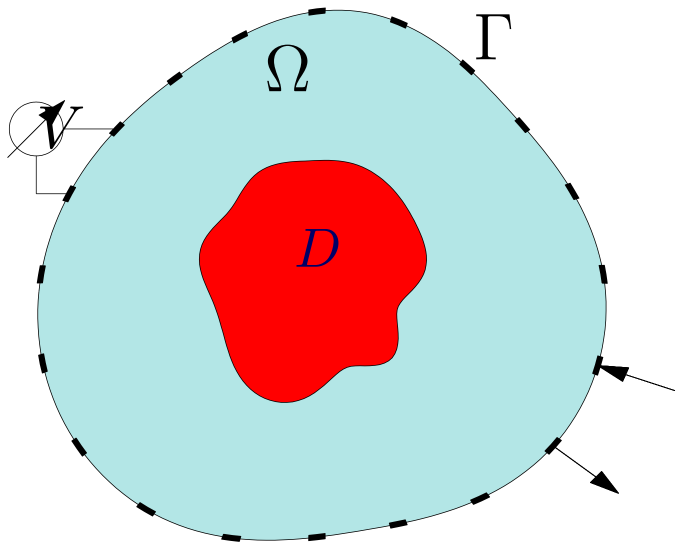
- self-adjoint and positive
- isomorphism from $H_\diamond^{-1/2}(\Gamma)$ onto $H_\diamond^{1/2}(\Gamma)$
- Hilbert-Schmidt operator (Hilbert space structure !)

---

given data : $\tilde{\Lambda} \approx \Lambda(\sigma)$

The Goal

Find all *discontinuities* of the conductivity σ



$$\sigma(x) = \begin{cases} 1 & \text{in } \Omega \setminus D \\ \kappa(x) < 1 & \text{in } D \end{cases}$$

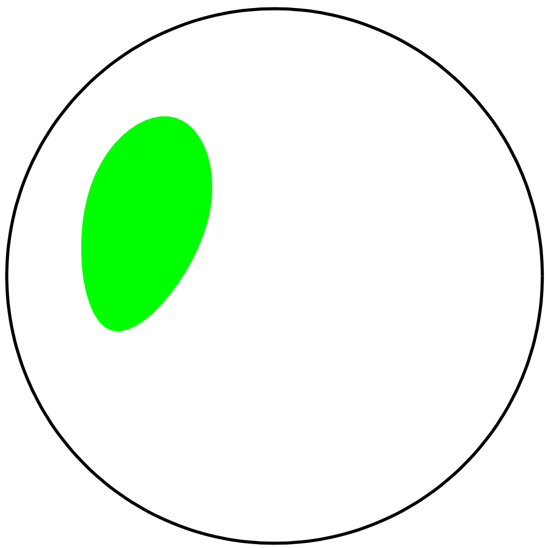
- σ is uniquely determined (Astala, Päivärinta, 2003)
- the problem is severely ill-posed (Allesandrini, 1988)

Factorization Method

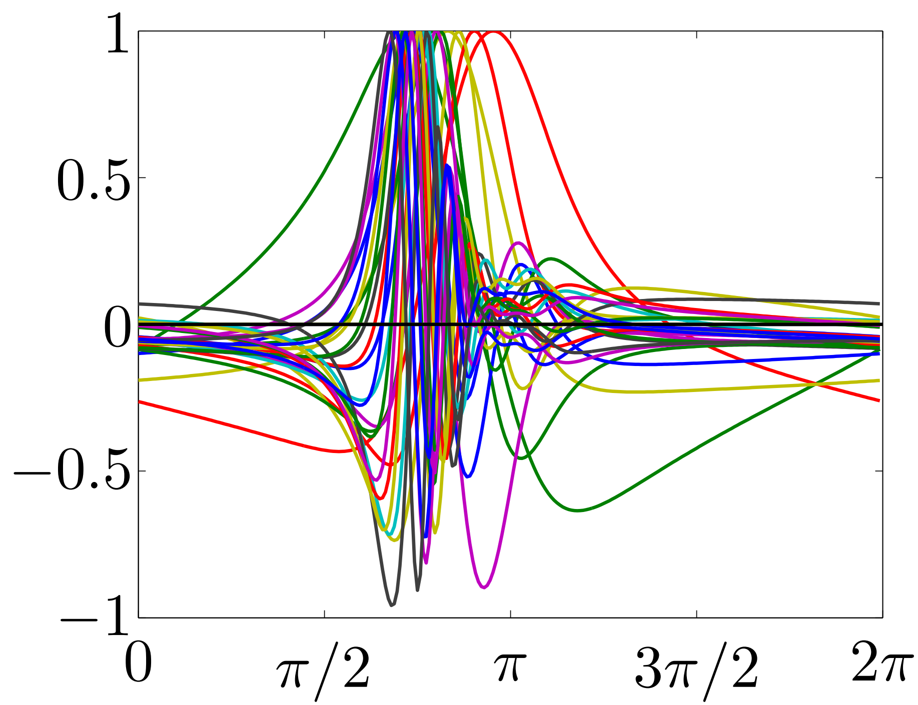
$$\tilde{\Lambda} - \Lambda = LFL'$$

The Range Space

Consider the differences in the boundary potentials :



$\rightsquigarrow$



What kind of information is in there ?

notation: $\tilde{\Lambda} = \Lambda(\sigma), \quad \Lambda = \Lambda(1)$

The Crucial Lemma

- Factorization $\tilde{\Lambda} - \Lambda = LFL'$:

$$L : \begin{cases} H_\diamond^{-1/2}(\partial D) \to H_\diamond^{1/2}(\Gamma)\,, \\ \varphi \mapsto w|_\Gamma \end{cases} \qquad \text{where} \qquad \begin{aligned} \Delta w &= 0 && \text{in } \Omega \setminus D\,, \\ \frac{\partial w}{\partial \nu} &= \begin{cases} 0 & \text{on } \Gamma\,, \\ \varphi & \text{on } \partial D \end{cases} \end{aligned}$$

- Obviously holds $\mathcal{R}(\tilde{\Lambda} - \Lambda) \subset \mathcal{R}(L)$:

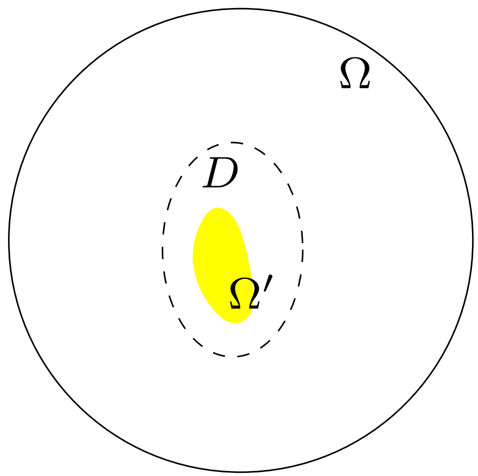
$$h = (\tilde{\Lambda} - \Lambda) f \qquad \rightsquigarrow \qquad h = v|_\Gamma\,, \quad v = \tilde{u} - u\,,$$

and v is a harmonic function in $\Omega \setminus D$ with

$$\frac{\partial v}{\partial \nu} = \frac{\partial \tilde{u}}{\partial u} - \frac{\partial u}{\partial \nu} = f - f = 0 \qquad \text{on } \Gamma$$

The Range of $\tilde{\Lambda} - \Lambda$

Assumption: $\Omega \setminus D$ can be reflected completely into D



Let $(\Omega \setminus D)'$ be the reflected set, and $\Omega' = D \setminus (\Omega \setminus D)'$ be the coloured set in the sketch

Theorem: $\mathcal{R}(\tilde{\Lambda} - \Lambda)$ is the set of traces on Γ of all harmonic functions

$$v \in H^1_\diamond(\Omega \setminus \Omega') \quad \text{with} \quad \frac{\partial v}{\partial \nu}\Big|_\Gamma = 0$$

Main Result

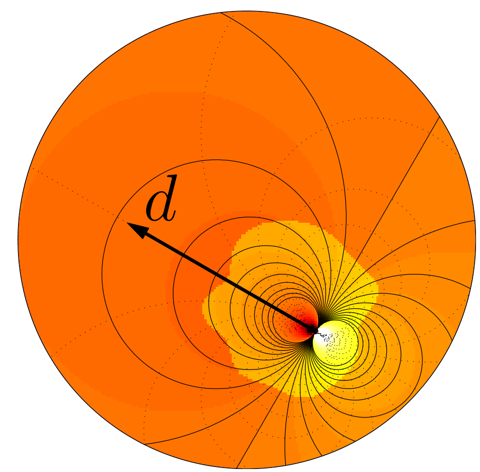
Brühl, H., 1999:

Not $\mathcal{R}(\tilde{\Lambda} - \Lambda)$, but the somewhat larger space $\mathcal{R}((\tilde{\Lambda} - \Lambda)^{1/2})$ is the correct one, as the latter one coincides with $\mathcal{R}(L)$

Corollary: The boundary values $h_{z,d}$ of a (modified) dipole potential belong to $\mathcal{R}((\tilde{\Lambda} - \Lambda)^{1/2})$, if and only if $z \in D$

for the unit circle:

$$h_{z,d}(x) = d \cdot \operatorname{grad} N_z(x) \; = \; \frac{1}{\pi} \, \frac{(z - x) \cdot d}{|z - x|^2}$$



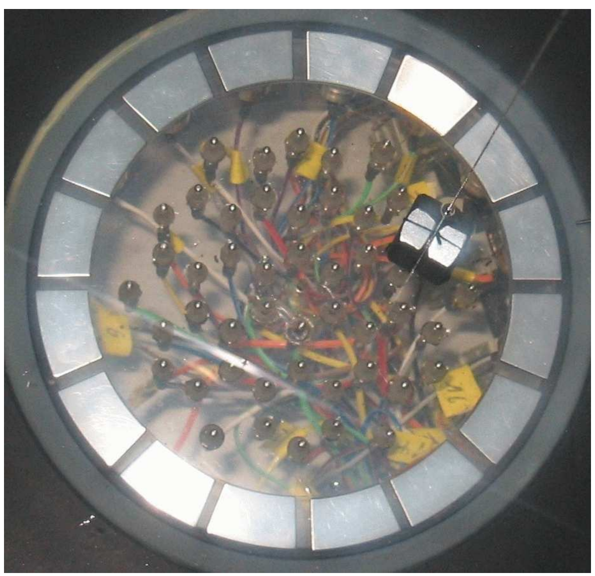
Applications

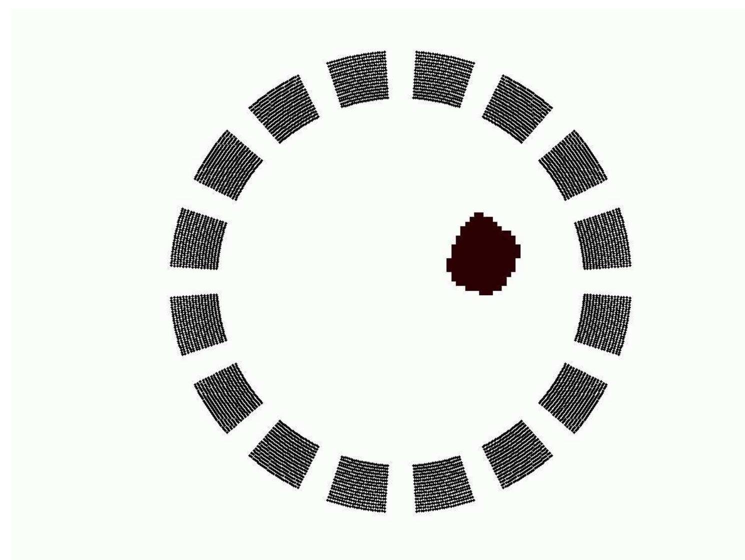
- Impedance tomography for mammography
- Impedance tomography in the half space
- Nondestructive testing of materials
- Detection of land mines

Mammography

Mainz system for mammography:

a typical reconstruction (AZZOUZ, H., OESTERLEIN, SCHAPPEL, 2006):





Half Space Geometry

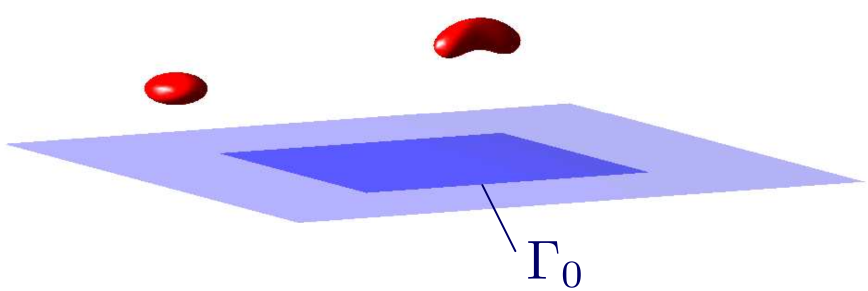
The half space is of particular interest for some applications (e.g., in geophysics)

Example: $\Omega = \mathbb{R}^3_+$ with $x = (\xi, \eta, \zeta)$ and $\zeta > 0$

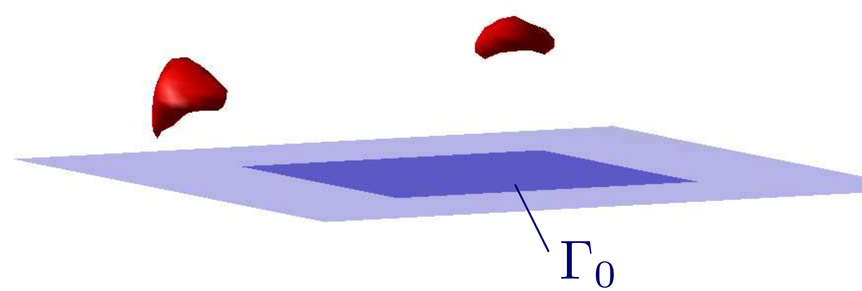
boundary: $\Gamma = \{\zeta = 0\}$, measurements: $\Gamma_0 = [-1, 1]^2 \subset \Gamma$

a typical reconstruction (H., Schappel, 2006) :

original:



reconstruction:

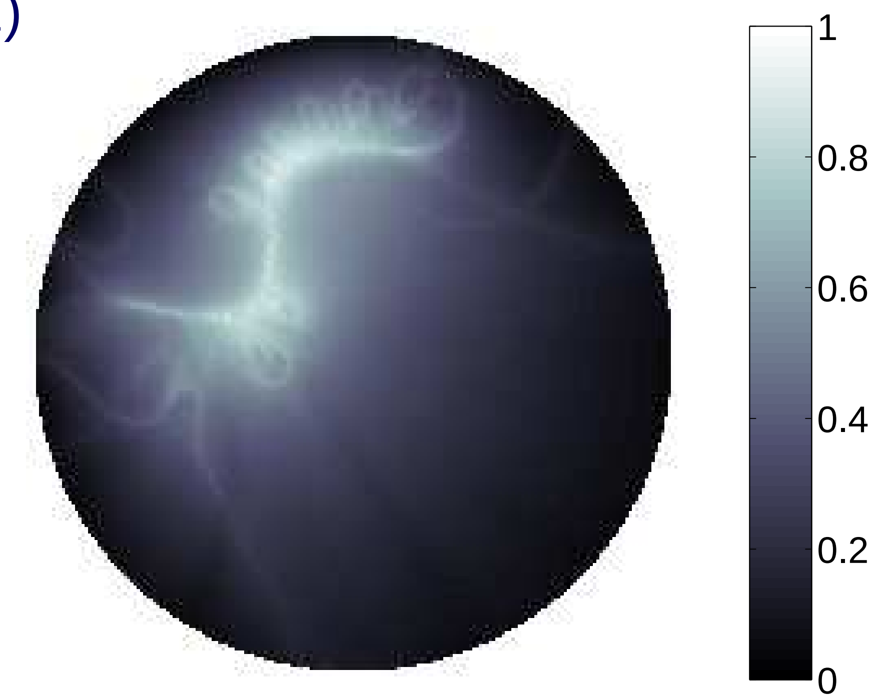


Nondestructive Testing

Investigation of a homogeneous conductor for (insulating) cracks

a typical reconstruction:

(Brühl, H., Pidcock, 2001)

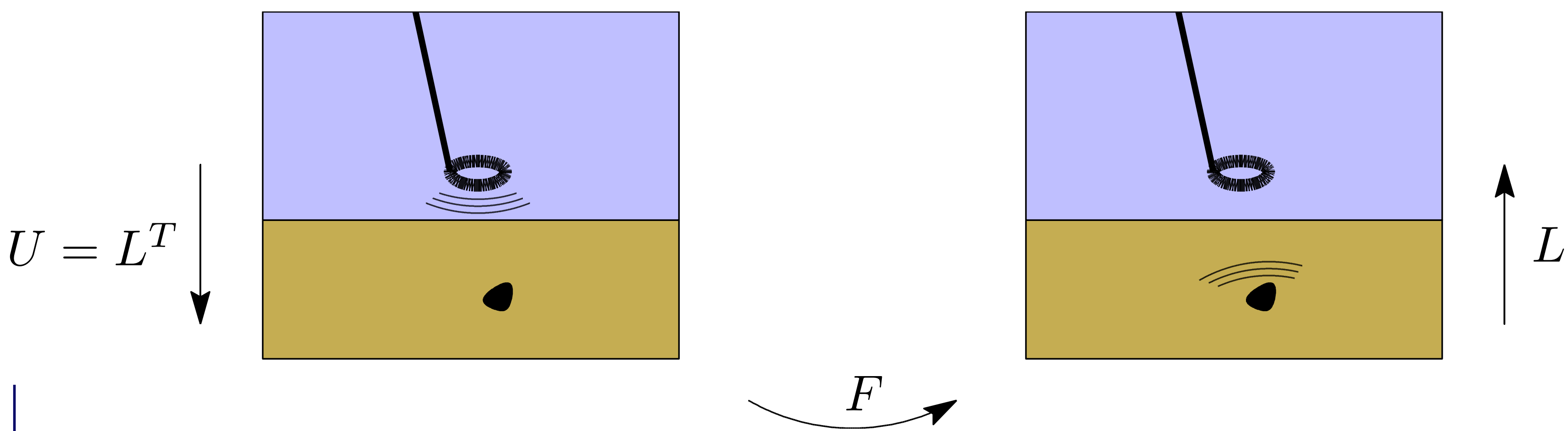


Detection of Land Mines

Interdisciplinary BMBF project:

Metal detectors for Humanitarian Demining:
Development potentials for data analysis and measurement techniques

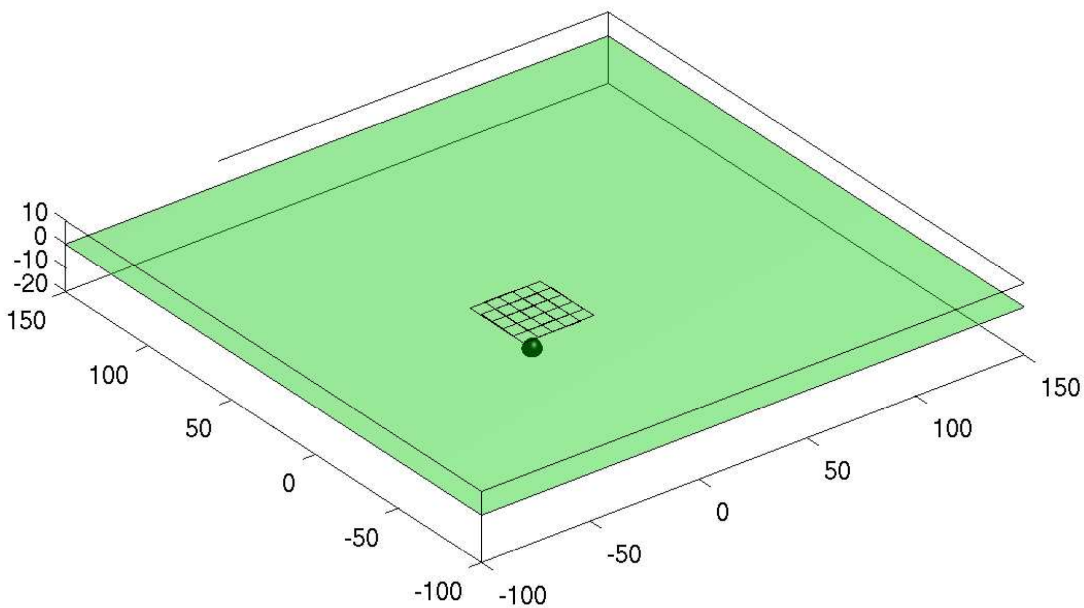
extension of the factorization method for the full Maxwell equations in a layered (or even more complicated) background

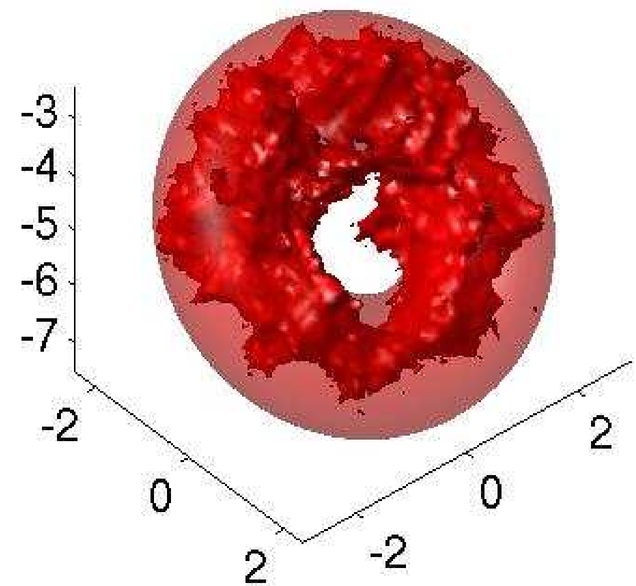


Detection of Land Mines

Multistatic (6×6) arrangement of commercial off-the-shelf metal detectors:

Example: reconstruction of a torus with a diameter of 6 cm and a height of 2 cm, placed 10 cm below the ground (wave length ≈ 300 km)





GEBAUER, H., KIRSCH, MUNIZ, SCHNEIDER, 2005

Implementation

$$z \in D \qquad \text{iff} \qquad h_{z,d} \in \mathcal{R}\big((\tilde{\Lambda} - \Lambda)^{1/2}\big)$$

Picard Criterion

$$z \in D \qquad \text{iff} \qquad h_{z,d} \in \mathcal{R}((\tilde{\Lambda} - \Lambda)^{1/2})$$

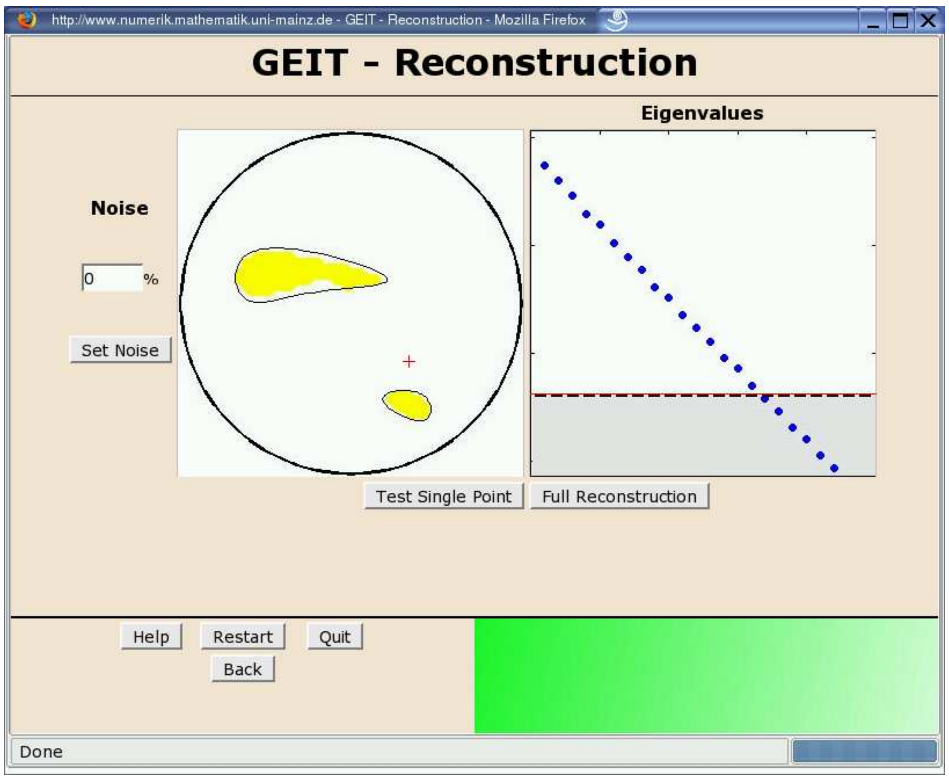
spectral decomposition: $(\tilde{\Lambda} - \Lambda)v_j = \lambda_j v_j \,, \qquad j = 1, 2, \ldots$

$$z \in D \qquad \text{iff} \qquad \sum_{j=1}^{\infty} \frac{\langle v_j, h_{z,d} \rangle^2}{\lambda_j} < \infty$$

Interactive Tool

Our algorithm is set up for interactive numerical experiments on the web

http://numerik.mathematik.uni-mainz.de/geit



BRÜHL, GEBAUER, 2002

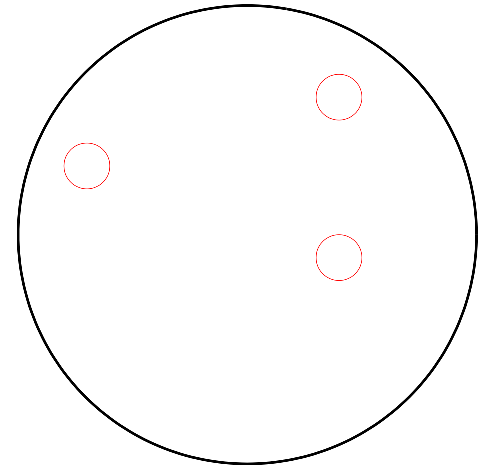
A MUSIC-Type Algorithm

MUSIC-Algorithm (for inverse scattering problems):

Determine a finite number of scatterers as fictitious point sources

DEVANEY, CHENEY, KIRSCH, ...

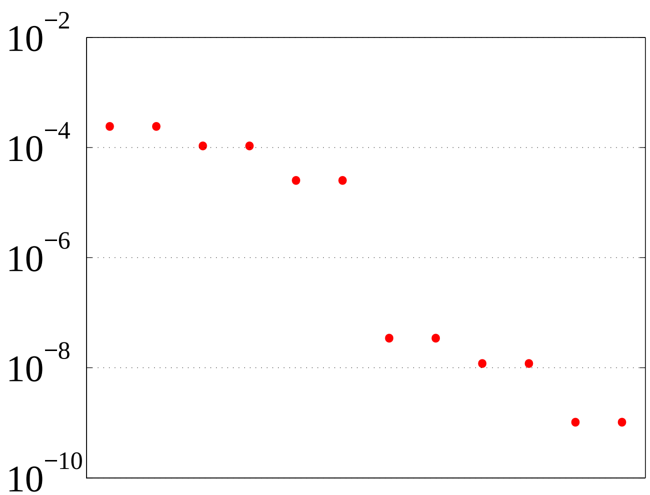
Impedance Tomography

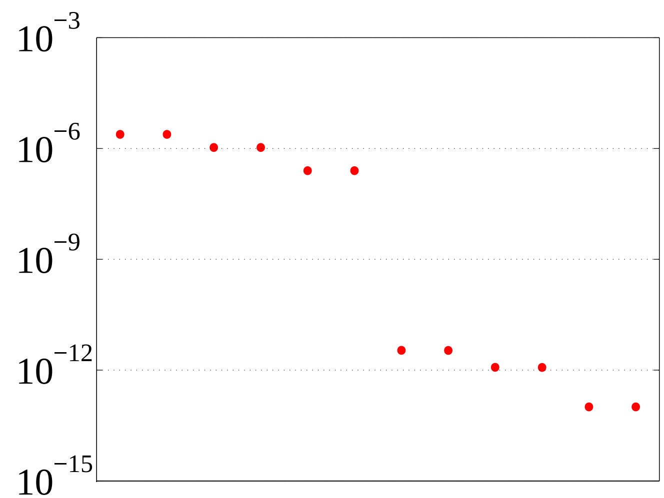


Observation (BRÜHL, H., VOGELIUS, 2002,
AMMARI ET AL, 2004, ...):

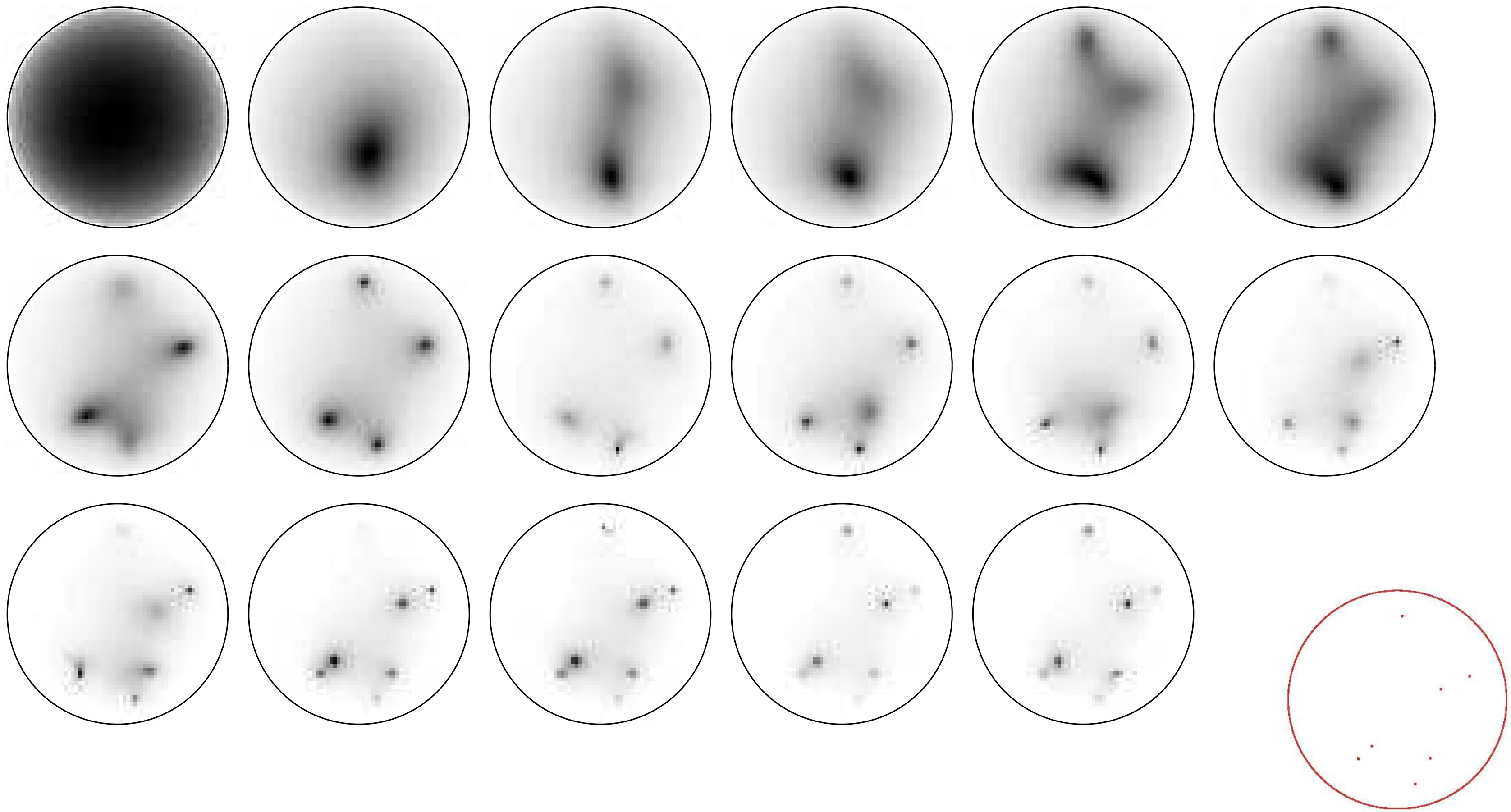
Given p "small" inclusions, the set $\mathcal{R}(\tilde{\Lambda} - \Lambda)$

- has dimension $2p$, essentially, and
- is spanned by dipoles placed in the centers of the inclusions



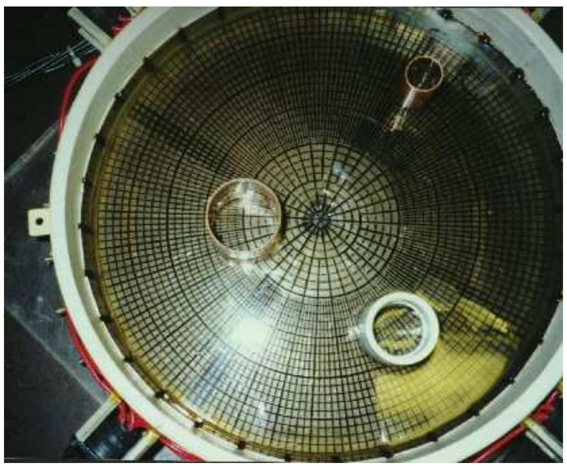


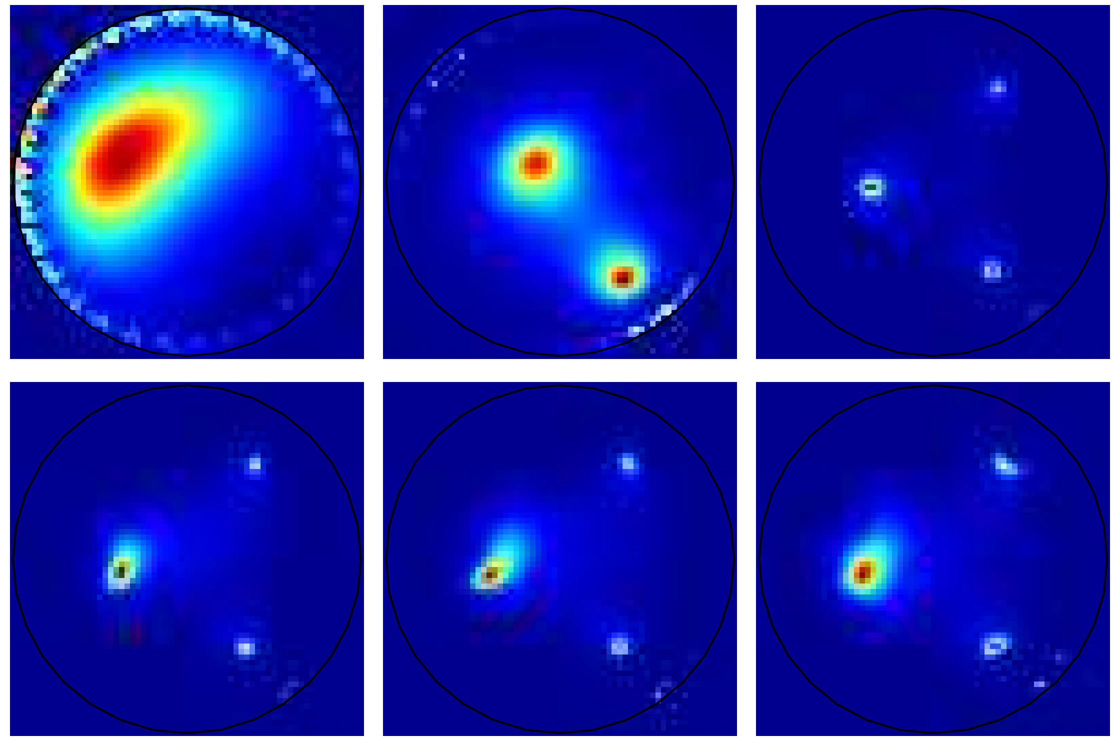
MUSIC



from BRÜHL, H., VOGELIUS, 2002

An Example with Real Data



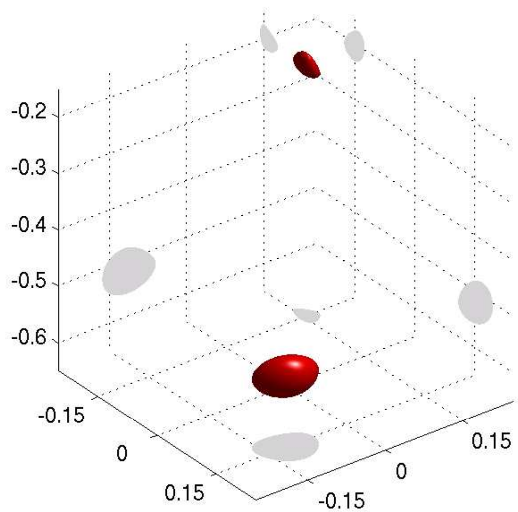


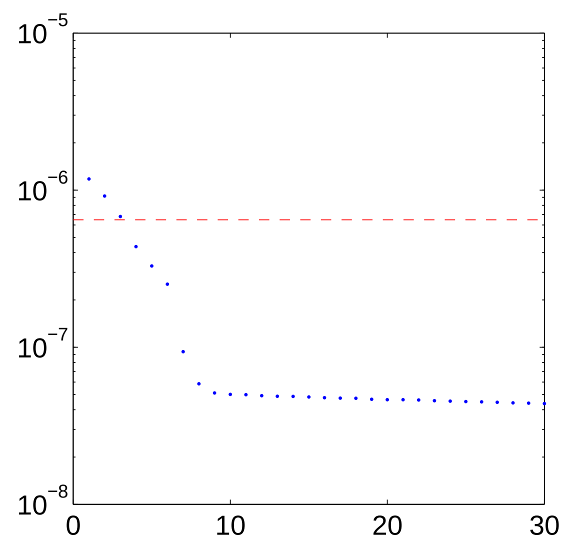
data have been kindly provided by RPI

Detection of Land Mines

Work in progress: Extend this asymptotic result to the mine problem
AMMARI, GRIESMAIER, H., 2006, GRIESMAIER, 2007

a typical reconstruction (from GRIESMAIER, 2007) :





wave number: $k = 4.2 \cdot 10^{-4}\,\mathrm{m}^{-1}$