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AN INVERSE BACKSCATTER PROBLEM FOR ELECTRIC IMPEDANCE TOMOGRAPHY*

MARTIN HANKE[†], NUUTTI HYVÖNEN[‡], AND STEFANIE REUSSWIG[†]

Abstract. We consider a variant of (two dimensional) electric impedance tomography with very sparse data that resemble so-called backscatter data in inverse scattering. Such data arise in practice if the same single pair of electrodes is used to drive currents and measure voltage differences, subsequently at various neighboring locations on the boundary of the object to be illuminated. We prove that these data uniquely determine an insulating cavity within the object.

Key words. electric impedance tomography, backscatter, inclusions, uniqueness theorem, Schwarzian derivative

AMS subject classifications. 35R30, 65N21

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1. Introduction.

In the context of the scattering of waves, backscatter data refer to measurements of the scattered wave that are taken at the very same location where the corresponding incident wave was generated. Intuitively, one may argue that within the entire scattered field, the backscatter portion carries a most significant amount of information about the scattering object, in particular when the length scale of the object is considerably above the resonance region. In fact, if backscatter data were available for a range of frequencies, then generic uniqueness results would be available that support this claim, even in two space dimensions; cf., e.g., [4, 16, 18, 20], and references therein. Much less is known, though, concerning the question whether time-harmonic backscatter data with *fixed* frequency do suffice to locate and possibly reconstruct a scattering object; see [7, 8, 15] for some partial results. This is a particularly miserable situation as there are quite a number of applications where backscatter data may provide the only realistic and reasonably accurate piece of information about the scattering object, not to mention the aspect of efficiency when a single sensor can replace a collection of sensors that are necessary to measure the scattered field in all possible directions.

In this paper, we consider the backscatter problem for the Laplace equation in two space dimensions. This may be viewed as a simple model for the low frequency regime in time-harmonic electromagnetic scattering, but a more obvious application—which we will adopt as the theoretical framework of our results—is the impedance tomography problem which has been raised in a seminal paper by Caldéron [3] back in 1980. In impedance tomography one attempts to reconstruct the conductivity distribution within a certain object from electrostatic potentials on the boundary of the object for all possible boundary currents. In mathematical terms this amounts

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[†]Institut für Mathematik, Johannes Gutenberg-Universität Mainz, 55099 Mainz, Germany (hanke@math.uni-mainz.de, reusswig@math.uni-mainz.de). The work of the third author was supported by the Deutsche Forschungsgemeinschaft (DFG) under grant HA 2121/6-1.

[‡]Institute of Mathematics, Helsinki University of Technology, FI-02015 HUT, Finland (nuutti.hyvonen@hut.fi). The work of this author was supported by the Academy of Finland (project 115013), the Finnish Funding Agency for Technology and Innovation (project 40084/06), the Finnish Cultural Foundation, and the Finnish Foundation for Technology Promotion.

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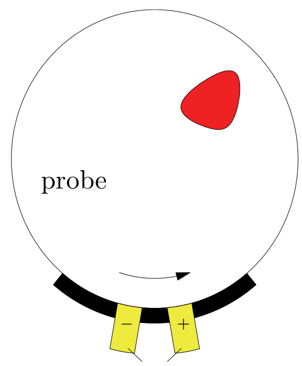

FIG. 1.1. *Sketch of a possible experimental setup for nondestructive testing: Between two successive measurements the probe is rotated by a fixed angle.*

to the full so-called Neumann-to-Dirichlet map of the associated elliptic differential operator as input data for the inverse problem.

In contrast to this standard formulation, our notion of backscattering gives rise to a very peculiar and restricted set of data—somewhat related to the main diagonal of the Neumann-to-Dirichlet map. As in the scattering context above, however, this piece of data has an obvious practical relevance, because measurements of this type occur if a single pair of nearby electrodes is used to drive currents and simultaneously measure the resulting voltage difference at various points on the boundary of the object; see Figure 1.1 for a possible experimental setup in nondestructive testing.

For such a sparse set of data it is certainly not possible to reconstruct a general conductivity distribution within the object. Therefore we restrict our attention to the following inverse problem: Given a two dimensional homogeneous and conducting object with known conductivity and an unknown embedded insulating cavity, can we determine the cavity from the corresponding backscatter data? The answer turns out to be "yes": We will prove below that a single cavity is uniquely determined by this piece of data; in fact, even a subset of these data from a relatively open portion of the boundary suffices for this purpose.

We predominantly use complex variable tools to establish these results, most notably the Riemann mapping theorem for doubly connected regions. In this respect, our approach is similar to the one by Kress [14] for the impedance tomography problem with one fixed input current, although both the problem setting as well as the details of the arguments are clearly different from those in [14]. As the Riemann mapping theorem is intimately connected to the Laplace equation in two space dimensions, our methods may not be suited for generalizations to the corresponding problems in the scattering of waves. Still we hope that this uniqueness result may initiate and possibly fertilize new research on these important problems.

For the sake of completeness, we mention that it is well known that less data than the full Neumann-to-Dirichlet map suffice to uniquely determine inhomogeneities in a homogeneous background by impedance tomography techniques. For example, the aforementioned setting of [14] with only one boundary current and the resulting potential (measured on an open subset of the boundary) as data are enough to identify an insulating cavity; cf., e.g., Beretta and Vessella [2]. More data appear to be necessary, however, when the inhomogeneity has a nonzero conductivity; see, for

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example, [1, 5, 12, 24]. In any case, take note that none of the earlier results apply to the setting of this work.

The outline of this paper is as follows. In section 2 we specify what we mean by backscatter data in impedance tomography, and we provide analytical and numerical examples for illustration. Subsequently, in section 3, we establish a connection between the backscatter data and the boundary map of a conformal transformation that is associated with the cavity. It then remains to prove that this connection uniquely determines the conformal map and the cavity, which amounts to investigating a certain boundary value problem for a nonlinear second order ordinary differential equation; see section 4. In section 5 we outline possible generalizations of our findings.

Finally, in the appendix, we justify our notion of backscatter data by showing—on the grounds of the so-called gap electrode model (cf., e.g., Isaacson and Cheney [11])—that these data are first order approximations to real-world electrode measurements.

2. Problem setting. Let D denote the open unit disk, Ω a simply connected domain with C^2-boundary $\Gamma = \partial\Omega \subset D$, and $\sigma : D \to \mathbb{R}^+$ a piecewise constant positive conductivity of the form

$$\sigma(x) \;=\; \begin{cases} 1\,, & x \in D \setminus \overline{\Omega}\,, \\ \kappa\,, & x \in \Omega\,. \end{cases} \tag{2.1}$$

For the moment we restrict $\kappa \neq 1$ to be positive, but we shall see (cf. (2.16)) that the setting of the problem readily extends to the degenerate case where $\kappa = 0$, i.e., where Ω is an insulating inclusion. In fact, it will be this latter case that will move into the focus of our attention from section 3 onwards.

In what follows, we denote by ν and τ, respectively, the exterior unit normal and the unit tangent at a point on the boundary of a given domain in $\mathbb{R}^2$. (The tangent is obtained by a counterclockwise $\pi/2$-rotation of ν.) For the unit circle we parameterize the points on T by $x_t = (\cos t, \sin t)$, $0 \le t < 2\pi$, and sometimes write $x_t^\perp = (-\sin t, \cos t)$ instead of τ for the tangent of T at x_t.

We now fix $\theta \in [0, 2\pi)$ and consider the Neumann boundary value problem

$$\nabla \cdot (\sigma \nabla u) \;=\; 0 \quad \text{in } D\,, \qquad \frac{\partial}{\partial \nu} u \;=\; \frac{\partial}{\partial \tau}\, \delta(\,\cdot\, - x_\theta)\,. \tag{2.2}$$

Here, the right-hand side $f = \partial/\partial\tau\, \delta(\,\cdot\, - x_\theta)$ of the Neumann boundary condition has to be understood as an element of $H^{-3/2-\varepsilon}(T)$ for some $\varepsilon > 0$, defined by virtue of

$$\langle\, f, v \,\rangle_{H^{-3/2-\varepsilon}(T) \times H^{3/2+\varepsilon}(T)} \;=\; -\frac{\partial v}{\partial s}\,(x_\theta) \qquad \text{for every } \; v \in H^{3/2+\varepsilon}(T)\,, \tag{2.3}$$

where s is the arc length parameter (i.e., the polar angle) of T. Problem (2.2) is known to have a unique solution $u \in H^{-\varepsilon}(D)$ up to the choice of the ground level of the potential, i.e., up to an additive constant; see Lions and Magenes [17]. We choose this constant in such a way that

$$\int_T u \,\mathrm{d}s \;=\; 0\,. \tag{2.4}$$

The solution u fulfills the differential equation in the sense of distributions, and the Neumann boundary data of u coincide with f of (2.3) in the sense of the appropriate trace theorem; cf. [17, Chapter 2, Remark 7.2]. Finally, the normalization

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constraint (2.4) is short for

$$\langle\, u|_T, 1\,\rangle_{H^{-1/2-\varepsilon}(T)\times H^{1/2+\varepsilon}(T)} = 0\,,$$

where $u|_T \in H^{-1/2-\varepsilon}(T)$ has to be interpreted in the context of the appropriate trace theorem (cf. [17, Chapter 2, Theorem 6.5]); as we will see below, though, in our context (2.4) has a proper (classical) meaning because $u|_T$ is a C^∞-function on $T \setminus \{x_\theta\}$ and the integral in (2.4) exists as a Cauchy principal value. (Bear in mind that u is smooth away from x_θ and Γ due to the regularity theory of elliptic partial differential equations; cf. [17].)

It follows from Weyl's lemma that the first equation of (2.2) is equivalent to asking that u be harmonic in $D \setminus \Gamma$ and satisfy the jump conditions

$$u^+|_\Gamma - u^-|_\Gamma \,=\, 0 \qquad \text{and} \qquad \frac{\partial}{\partial\nu}\, u^+\Big|_\Gamma - \kappa\,\frac{\partial}{\partial\nu}\, u^-\Big|_\Gamma \,=\, 0 \tag{2.5}$$

at the interface Γ, where $u^+ = u|_{D\setminus\overline{\Omega}}$ and $u^- = u|_\Omega$. Moreover, from the aforementioned interpretation of the Neumann boundary condition we conclude that u is the limit in $H^{-\varepsilon}(\Omega)$ of the solutions $u_n \in H^1(\Omega)$, $n \in \mathbb{N}$, of the perturbed boundary value problems

$$\nabla\cdot(\sigma\nabla u_n) \,=\, 0 \quad \text{in } D\,, \qquad \frac{\partial}{\partial\nu}u_n \,=\, f_n \quad \text{on } T\,, \qquad \int_T u_n \,\mathrm{d}s \,=\, 0\,, \tag{2.6}$$

where

$$(f_n)_n \subset \mathcal{L}^2_\diamond(T) \,=\, \Big\{ f \in \mathcal{L}^2(T) \,:\, \int_T f \,\mathrm{d}s \,=\, 0 \Big\}$$

is any sequence that converges to f of (2.3) in $H^{-3/2-\varepsilon}(T)$ as $n \to \infty$, i.e.,

$$\|f_n - f\|_{H^{-3/2-\varepsilon}(T)} = \sup_{\|v\|_{H^{3/2+\varepsilon}}=1} \left| \int_T f_n v \,\mathrm{d}s \,+\, \frac{\partial v}{\partial s}\,(x_\theta) \right| \to 0\,. \tag{2.7}$$

Note that such a sequence exists since $\mathcal{L}^2(T)$, or even $C^\infty(T)$, is dense in $H^{-3/2-\varepsilon}(T)$; see, e.g., [17, Chapter 1, section 7.3].

Fortunately, we can make use of a rather explicit representation for the solution u of (2.2). To derive this formula we consider first the special case where $\kappa = 1$, so that the differential operator in (2.2) reduces to the Laplacian. In this case the solution u_n of (2.6) can be written as integral

$$u_n(z) \,=\, \int_T N(z, x_t) f_n(x_t)\,\mathrm{d}s(x_t)\,, \qquad z \in D\,, \tag{2.8}$$

where N is the Neumann function for the Laplacian in D, i.e.,

$$N(z,x) \,=\, \begin{cases} -\dfrac{1}{2\pi}\left(\log|z-x| \,+\, \log\left|\dfrac{z}{|z|} - |z|x\right|\right), & z \neq 0\,,\\[2ex] -\dfrac{1}{2\pi}\,\log|x|\,, & z = 0\,. \end{cases} \tag{2.9}$$

Because of (2.7) we conclude that u_n converges to

$$u_0(z) \,=\, -\frac{\partial}{\partial_x\tau}N(z,x_\theta) \,=\, -\frac{1}{\pi}\,\frac{z\cdot x_\theta^\perp}{|z-x_\theta|^2}\,, \qquad z \neq x_\theta\,. \tag{2.10}$$

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Note that u_0 has a first order pole at $z = x_\theta$ so that its trace does not belong to $\mathcal{L}^1(T)$; more precisely, we have

$$u_0(x_t) \;=\; -\frac{1}{2\pi}\cot\frac{t-\theta}{2}\,, \qquad x_t \neq x_\theta\,, \tag{2.11}$$

which reveals that u_0 satisfies (2.4) in the sense of a principal value integral.

Next we turn to the case $\kappa \neq 1$ and define

$$w(z) \;=\; \int_\Gamma N(z,x)\psi(x)\,\mathrm{d}s(x)\,, \qquad z \in D\setminus\Gamma\,, \tag{2.12}$$

as a single layer potential over Γ with density

$$\psi \in C_\diamond(\Gamma) \;=\; \left\{\psi \in C(\Gamma)\,:\, \int_\Gamma \psi\,\mathrm{d}s = 0\right\}.$$

Then $u_0 + w$ is harmonic in $D\setminus\Gamma$, it satisfies the first jump condition of (2.5) and the boundary conditions of (2.2) and (2.4) (the latter as a principal value integral), and its flux on Γ is given by

$$\frac{\partial}{\partial\nu}(u_0+w)^\pm\Big|_\Gamma \;=\; \frac{\partial u_0}{\partial\nu}\Big|_\Gamma \;+\; K_\Gamma^*\psi \;\mp\; \frac{1}{2}\,\psi\,,$$

where

$$K_\Gamma^*\psi(z) \;=\; \int_\Gamma \frac{\partial}{\partial_z\nu}N(z,x)\psi(x)\,\mathrm{d}s(x)\,, \qquad z\in\Gamma\,,$$

is a compact operator from $C_\diamond(\Gamma)\to C_\diamond(\Gamma)$ (cf., e.g., Kress [13]). Accordingly, if we take $\psi \in C_\diamond(\Gamma)$ to be the solution of the integral equation

$$\left(I \,-\, 2\,\frac{1-\kappa}{1+\kappa}\,K_\Gamma^*\right)\psi \;=\; 2\,\frac{1-\kappa}{1+\kappa}\,\frac{\partial u_0}{\partial\nu}\Big|_\Gamma\,, \tag{2.13}$$

then $u = u_0 + w$, with u_0 of (2.10) and w of (2.12), also satisfies the second jump condition of (2.5), and is thus the solution of the basic equation (2.2). Note that $\partial u_0/\partial\nu$ has vanishing mean on Γ by virtue of (2.2) and Green's theorem. The existence of a unique solution $\psi \in C_\diamond(\Gamma)$ to (2.13) follows from the Fredholm alternative; cf., e.g., Schappel [22, Lemma 3.17], where this argument has been worked out for the Neumann function corresponding to the half space.

The potential u_0 of (2.10) will play a central role as *reference potential* in our construction: It satisfies the boundary value problem

$$\begin{gathered}\Delta u_0 \;=\; 0 \quad \text{in } D\,,\\ \frac{\partial}{\partial\nu}\,u_0 \;=\; \frac{\partial}{\partial\tau}\,\delta(\,\cdot\,-x_\theta) \quad \text{on } T\,, \qquad \int_T u_0\,\mathrm{d}s \;=\; 0\,,\end{gathered} \tag{2.14}$$

corresponding to a homogeneous body where $\sigma = 1$ throughout D. The so-called *relative potential*

$$w = u - u_0$$

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is given by (2.12) and is smooth in the exterior of $\overline{\Omega}$; in particular, the quantity

$$b(\theta) \;=\; -\left(\frac{\partial}{\partial\tau}\, w|_T\right)(x_\theta) \tag{2.15}$$

is well defined.

Finally, we mention that the previous considerations extend to the degenerate case where $\kappa = 0$, i.e., where the inclusion in Ω is insulating. In this case, (2.2) and (2.4) have to be replaced by

$$\begin{aligned} \Delta u \;&=\; 0 \quad \text{in } D\setminus\overline{\Omega}\,, \qquad & \frac{\partial}{\partial\nu}\, u \;&=\; 0 \quad \text{on } \Gamma\,,\\ \frac{\partial}{\partial\nu}\, u \;&=\; \frac{\partial}{\partial\tau}\,\delta(\,\cdot\, - x_\theta) \quad \text{on } T\,, \qquad & \int_T u \,\mathrm{d}s \;&=\; 0\,, \end{aligned} \tag{2.16}$$

the solution of which belongs to $H^{-\varepsilon}(D\setminus\overline{\Omega})$; take note that this solution is smooth away from $x_\theta \in T$ and, in particular, satisfies the Neumann condition on Γ in the classical sense (cf., e.g., [17]). The relative potential $w = u - u_0$ still satisfies (2.12) in $D\setminus\overline{\Omega}$, where ψ is the solution of (2.13) with $\kappa = 0$.

Note that all the aforementioned potentials depend on the parameter θ via the boundary condition in (2.2), although we have refrained from emphasizing this dependency so far. In what follows, however, we are interested in the *function* $b : [0, 2\pi) \to \mathbb{R}$ that is defined via (2.15) for all θ in this interval. We call this function the *backscatter data* associated with the inclusion Ω. One can interpret $b(\theta)$ as the induced relative potential between two infinitesimal points centered around a receiver position x_θ, when a dipole-type current is applied at the very same location $x_\theta \in T$; we will argue in the appendix that $b(\theta)$ determines the leading order term of real-life measurements by two electrodes attached very close to each other.

Example 2.1. For a concentric circle $\Omega = B_R$ of radius $R > 0$ the backscatter data are easily seen to be independent of θ. They can be evaluated, for example, from a Fourier series expansion of u and u_0. Without loss of generality we consider only the location $\theta = 0$ for the dipole. In this case we have, for $x = rx_t$ with $R < r < 1$ and $0 \le t < 2\pi$, that

$$u_0(x) \;=\; -\frac{1}{\pi}\sum_{k=1}^{\infty} r^k \sin kt$$

and

$$u(x) \;=\; -\frac{1}{\pi}\sum_{k=1}^{\infty}\bigl(\gamma_k r^k \;+\; (\gamma_k - 1) r^{-k}\bigr)\sin kt\,,$$

where

$$\gamma_k \;=\; \frac{1}{1 - \frac{1-\kappa}{1+\kappa} R^{2k}}\,. \tag{2.17}$$

It follows that

$$w(x) \;=\; -\frac{1}{\pi}\sum_{k=1}^{\infty}(\gamma_k - 1)(r^k + r^{-k})\sin kt\,,$$

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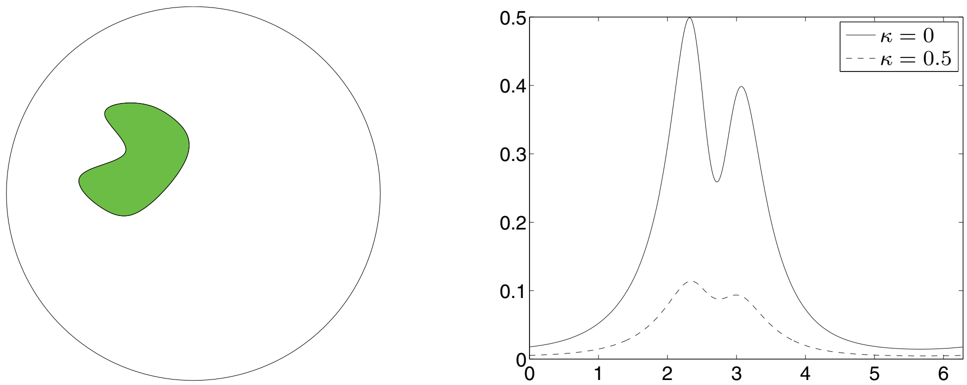

FIG. 2.1. *Numerical simulation: Backscatter data (right) for a nontrivial inclusion (left) and two different conductivities.*

and hence,

$$b(\theta) \;=\; \frac{2}{\pi}\sum_{k=1}^{\infty} k\,\frac{\frac{1-\kappa}{1+\kappa}R^{2k}}{1-\frac{1-\kappa}{1+\kappa}R^{2k}}\,. \tag{2.18}$$

We observe immediately from (2.18) that, in general, the domain Ω of the inclusion cannot be determined from the backscatter data if the conductivity within Ω is not known beforehand.

For example, if the domain Ω is known to correspond to an insulating object, then the right-hand side of (2.18) simplifies to

$$\beta_R \;=\; \frac{2}{\pi}\sum_{k=1}^{\infty} k\,\frac{R^{2k}}{1-R^{2k}} \;>\; 0\,. \tag{2.19}$$

Note that β_R is monotonically increasing with R and, in fact, increasing without bound as $R \to 1$. Thus, if the backscatter data $b(\theta)$ are positive and independent of θ, then there is a unique insulating disk centered at the origin that produces these data.

Example 2.2. Figure 2.1 shows the backscatter data corresponding to a nontrivial domain Ω strictly inside the unit disk for two different values κ of the conductivity within Ω. While the solid line in the right-hand plot corresponds to an insulating domain Ω, i.e., $\kappa = 0$, the conductivity has been set to $\kappa = 0.5$ for the backscatter data shown as a dashed line. These data have been computed numerically for each parameter θ separately, on the grounds of formula (2.12). Since it is not obvious from the figure, we remark that the two functions in the right-hand plot are not constant multiples of each other.

3. Backscatter data and conformal transformations. Motivated by the insight from Example 2.1, we restrict our attention in what follows to insulating inclusions that fill a (nonempty) simply connected C^2-domain Ω with $\overline{\Omega} \subset D$. Such sets will be called *cavities* further on. For every cavity Ω there exists a conformal map Φ, which takes $D \setminus \overline{\Omega}$ onto a concentric annulus $\{x \in D \,:\, R < |x| < 1\}$ (cf., e.g., Henrici [9, Theorem 5.10h]). The inner radius $R > 0$ is a characteristic number of the domain Ω. Since Φ maps T onto itself, it can be extended by reflection to

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a holomorphic function—in fact, a conformal map—in a certain neighborhood of T (cf. [9, Theorem 5.11b]), and hence, the associated map of the corresponding angular parameters of the boundary,

$$\varphi(t) \,=\, \arg\Phi(e^{\mathrm{i}t}) \,=\, \frac{1}{\mathrm{i}}\log\bigl(\Phi(e^{\mathrm{i}t})\bigr)\,, \qquad 0 \le t < 2\pi\,, \tag{3.1}$$

extends to an analytic function on the real line. We remark that

$$\varphi'(t) \,=\, \Phi'(e^{\mathrm{i}t})\, e^{\mathrm{i}(t-\varphi(t))} \,=\, |\Phi'(e^{\mathrm{i}t})| > 0\,, \tag{3.2}$$

because φ' is real and positive, as Φ leaves the orientation of T invariant. Moreover, Φ has a continuously differentiable extension which maps Γ bijectively onto ∂B_R (cf., e.g., Pommerenke [19, section 3.3]).

Proposition 3.1. *Let Ψ be the inverse of the conformal map Φ introduced above. Furthermore, let u be the solution of* (2.16)*, and let $\tilde{u}$ be the solution of the respective boundary value problem* (2.16) *where Ω and θ are replaced by B_R and $\tilde{\theta} = \varphi(\theta)$, respectively. Then there exists $c \in \mathbb{R}$ such that*

$$\tilde{u} \,=\, \frac{1}{\varphi'(\theta)}\, u \circ \Psi \,+\, c\,. \tag{3.3}$$

Proof. To begin with, note that $\tilde{v} = v \circ \Psi \in H^{3/2+\varepsilon}(T)$ for any $v \in H^{3/2+\varepsilon}(T)$ since $\Phi|_T$ is a C^∞-diffeomorphism of T onto itself. Furthermore,

$$\|\tilde{v}\|_{H^{3/2+\varepsilon}(T)} \,\le\, C\,\|v\|_{H^{3/2+\varepsilon}(T)}\,, \tag{3.4}$$

with $C = C(\varepsilon) > 0$ independent of $v \in H^{3/2+\varepsilon}(T)$; see, e.g., the definitions in [17, Chapter 1, section 7.3].

Let $(\tilde{f}_n)_n \subset \mathcal{L}^2_\diamond(T) \cap C^\infty(T)$ be any sequence converging to the tangential derivative of the delta distribution at $x_{\tilde{\theta}}$ in the topology of $H^{-3/2-\varepsilon}(T)$, and define

$$f_n \,=\, \varphi'(\theta)\bigl(\tilde{f}_n \circ \Phi\bigr)\,|\Phi'| \qquad \text{on } T\,.$$

By denoting the angular variable in the original domain by t and that in the annulus by $\tilde{t} = \varphi(t)$ in accordance with (3.1), we obtain

$$\int_T f_n v \,\mathrm{d}s(x_t) \,=\, \int_T \varphi'(\theta)\,\bigl(\tilde{f}_n \circ \Phi\bigr)\bigl(\tilde{v} \circ \Phi\bigr)\,|\Phi'|\,\mathrm{d}s(x_t) \,=\, \varphi'(\theta)\int_T \tilde{f}_n \tilde{v}\,\mathrm{d}s(x_{\tilde{t}})\,,$$

and

$$\Bigl(\frac{\partial}{\partial t}\,v\Bigr)\Big|_{x_\theta} \,=\, \varphi'(\theta)\,\Bigl(\frac{\partial}{\partial t}\,v\Bigr)\Big|_{x_\theta}\,|\Psi'(x_{\tilde{\theta}})| \,=\, \varphi'(\theta)\,\Bigl(\frac{\partial}{\partial \tilde{t}}\,\tilde{v}\Bigr)\Big|_{x_{\tilde{\theta}}}\,.$$

In particular,

$$\Bigl|\,\int_T f_n v\,\mathrm{d}s(x_t) \,+\, \Bigl(\frac{\partial}{\partial t}\,v\Bigr)\Big|_{x_\theta}\,\Bigr| \,=\, \varphi'(\theta)\Bigl|\,\int_T \tilde{f}_n\tilde{v}\,\mathrm{d}s(x_{\tilde{t}}) \,+\, \Bigl(\frac{\partial}{\partial \tilde{t}}\,\tilde{v}\Bigr)\Big|_{x_{\tilde{\theta}}}\,\Bigr| \tag{3.5}$$

Taking the supremum of (3.5) over $v \in H^{3/2+\varepsilon}(T)$ with $\|v\| = 1$, and using the convergence of $(\tilde{f}_n)_n$ together with (3.4), it follows that $(f_n)_n$ converges to the tangential derivative of the delta distribution at x_θ in the norm of $H^{-3/2-\varepsilon}(T)$ (cf. (2.7)).

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Now, let u_n be the solution of

$$
\begin{aligned}
\Delta u_n &= 0 \quad \text{in } D \setminus \overline{\Omega}\,, & \frac{\partial}{\partial \nu} u_n &= 0 \quad \text{on } \Gamma\,, \\
\frac{\partial}{\partial \nu} u_n &= f_n \quad \text{on } T\,, & \int_T u_n \,\mathrm{d}s &= 0\,,
\end{aligned}
\tag{3.6}
$$

i.e., the analogue of (2.6) for $\kappa = 0$. In particular, u_n converges to u in the topology of $H^{-\varepsilon}(D \setminus \overline{\Omega})$; see section 2. We define

$$
\tilde{u}_n \;=\; \frac{1}{\varphi'(\theta)}\, u_n \circ \Psi \,+\, c_n\,,
\tag{3.7}
$$

where

$$
c_n \;=\; -\,\frac{1}{2\pi\varphi'(\theta)} \int_T u_n |\Phi'| \,\mathrm{d}s
\tag{3.8}
$$

is such that $\int \tilde{u}_n \,\mathrm{d}s = 0$. Then $\tilde{u}_n$ is a harmonic function in $D \setminus \overline{B}_R$ and satisfies the no flux condition on ∂B_R. Moreover, on T there holds

$$
\frac{\partial}{\partial \nu}\, \tilde{u}_n(x_{\tilde{t}}) \;=\; \frac{1}{|\Phi'(x_t)\varphi'(\theta)|}\, \frac{\partial}{\partial \nu}\, u_n(x_t) \;=\; \tilde{f}_n(x_{\tilde{t}})\,.
$$

In other words, $\tilde{u}_n$ is the solution of (3.6) for $\Omega = B_R$ and f_n replaced by $\tilde{f}_n$, and thus it converges to $\tilde{u}$ in $H^{-\varepsilon}(D \setminus \overline{B}_R)$. On the other hand, from (3.7) and (3.8) it follows that

$$
\tilde{u}_n \;\to\; \frac{1}{\varphi'(\theta)}\, u \circ \Psi \,+\, c\,, \qquad n \to \infty\,,
\tag{3.9}
$$

in the topology of $H^{-\varepsilon}(D \setminus \overline{B}_R)$, with

$$
c \;=\; -\,\frac{1}{2\pi\varphi'(\theta)}\, \langle\, u, |\Phi'|\, \rangle_{H^{-1/2-\varepsilon}(T)\times H^{1/2+\varepsilon}(T)}\,;
$$

see the trace theorems in [17]. The right-hand side of (3.9) is, therefore, $\tilde{u}$, i.e., the solution of (2.16) with Ω replaced by B_R and θ replaced by $\tilde{\theta}$, as was to be shown. $\square$

Of special interest is the situation when Ω is a nonconcentric disk, the concentric case being treated in Example 2.1. In this case Φ is a Möbius transformation of the form (in complex variables)

$$
\Phi(z) = \frac{z - \zeta}{1 - \overline{\zeta} z}\,,
\tag{3.10}
$$

where $\zeta \in D$ is some fixed parameter; cf., e.g., Schinzinger and Laura [23, section 3.2.1]. Writing $\zeta = \rho e^{\mathrm{i}s}$ in polar coordinates, the boundary map (3.1) associated with this particular Φ is given by

$$
\varphi(t) \;=\; s \,+\, 2\arctan\left(\frac{1+\rho}{1-\rho}\, \tan\frac{t-s}{2}\right)
\tag{3.11}
$$

(up to multiples of 2π) with derivative

$$
\varphi'(t) \;=\; \frac{1-\rho^2}{1 - 2\rho\cos(t-s) + \rho^2}\,.
\tag{3.12}
$$

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PROPOSITION 3.2. *If $\Omega = B_r(x)$ is an insulating disk with center $x \neq 0$ and radius $r < 1 - |x|$, then the backscatter data associated with Ω are given by*

$$b(\theta) \,=\, \beta_R\, \varphi'(\theta)^2\,, \tag{3.13}$$

where φ' is as in (3.12) *with s being the polar angle of x,*

$$\rho \,=\, \frac{1+|x|^2-r^2}{2\,|x|} \,-\, \sqrt{\left(\frac{1+|x|^2-r^2}{2\,|x|}\right)^2 \,-\, 1}\,, \tag{3.14}$$

and β_R being given by (2.19) *with $R^2 = (\rho - |x|)/(\rho - \rho^2|x|)$.*

Proof. By Proposition 3.1 the solution u of (2.16) is given by

$$u \,=\, \varphi'(\theta)\,\tilde u \circ \Phi \,+\, c\,,$$

where $\tilde u$ is the solution of (2.16) with Ω replaced by the associated concentric disk and θ replaced by $\tilde\theta = \varphi(\theta)$. As Φ is a conformal map of the entire disk D, we can use the same proof to show that in this case we also have the identity

$$u_0 \,=\, \varphi'(\theta)\,\tilde u_0 \circ \Phi \,+\, c_0\,, \tag{3.15}$$

where u_0 is the reference potential and $\tilde u_0$ satisfies the boundary value problem (2.14) with θ replaced by $\tilde\theta = \varphi(\theta)$. It follows that the relative potential satisfies

$$w \,=\, \varphi'(\theta)\bigl(\tilde u \circ \Phi \,-\, \tilde u_0 \circ \Phi\bigr) \,+\, c' \,=\, \varphi'(\theta)\,\tilde w \circ \Phi \,+\, c'\,,$$

where $\tilde w$ is the corresponding relative potential for the concentric cavity B_R and the source/receiver position at $x_{\tilde\theta}$. Accordingly,

$$b(\theta) \,=\, -\left(\frac{\partial}{\partial t}\,w|_T\right)(x_\theta) \,=\, -\varphi'(\theta)\,\left(\frac{\partial}{\partial \tilde t}\,\tilde w|_T\right)(x_{\tilde\theta})\,|\Phi'(x_\theta)| \,=\, \varphi'(\theta)^2\beta_R\,,$$

the latter following from Example 2.1.

For the computation of the parameter $\zeta = \rho e^{\mathrm{i}s}$ in (3.10) and the radius R of B_R, see [23, section 3.2.1]. ∎

Unfortunately, the proof of this proposition does not extend to noncircular simply connected domains Ω with $\overline{\Omega} \subset D$, as the corresponding conformal map cannot be extended to a conformal map of the entire disk D, and hence, u_0 of (3.15) has no extension to a harmonic function in D, i.e., it does not solve (2.14). Nonetheless, in what follows we will see that we can still give an explicit formula for the backscatter data, given the corresponding number β_R and the boundary map φ.

THEOREM 3.3. *Let Ω be a cavity in D, and let φ be the boundary map* (3.1) *of the conformal transformation Φ that takes $D \setminus \overline{\Omega}$ onto $D \setminus \overline{B}_R$ for an appropriate $R \in (0,1)$. Then the associated backscatter data are given by*

$$b(\theta) \,=\, \beta_R\varphi'(\theta)^2 \,+\, \frac{1}{12\pi} \,-\, \frac{1}{12\pi}\,\varphi'(\theta)^2 \,+\, \frac{1}{4\pi}\,\frac{\varphi''(\theta)^2}{\varphi'(\theta)^2} \,-\, \frac{1}{6\pi}\,\frac{\varphi'''(\theta)}{\varphi'(\theta)}\,. \tag{3.16}$$

Proof. As before we fix $\theta \in [0, 2\pi)$ and denote by u the solution of (2.16). According to Proposition 3.1 the function $\tilde u$ of (3.3) solves the corresponding boundary value problem with Ω replaced by B_R and θ replaced by $\tilde\theta = \varphi(\theta)$. We also introduce

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the two reference potentials u_0 and $\tilde{u}_0$ corresponding to the homogeneous domain and the two source locations x_θ and $x_{\tilde{\theta}}$, respectively, as well as the two relative potentials

$$w \,=\, u - u_0 \qquad \text{and} \qquad \tilde{w} = \tilde{u} - \tilde{u}_0\,.$$

It follows that

$$\begin{aligned} w \,&=\, \varphi'(\theta)\,\tilde{u}\circ\Phi \,-\, u_0 \,+\, c \\ &=\, \varphi'(\theta)\,\tilde{u}\circ\Phi \,-\, \varphi'(\theta)\,\tilde{u}_0\circ\Phi \,+\, \varphi'(\theta)\,\tilde{u}_0\circ\Phi \,-\, u_0 \,+\, c \\ &=\, \varphi'(\theta)\,\tilde{w}\circ\Phi \,+\, \varphi'(\theta)\,\tilde{u}_0\circ\Phi \,-\, u_0 \,+\, c\,. \end{aligned}$$

Next, we introduce the difference of the traces of the two reference potential terms on T as a function of the polar angle $t \in [0, 2\pi)$, i.e.,

$$d(t) \,=\, \varphi'(\theta)\,\tilde{u}_0\big(\Phi(x_t)\big) \,-\, u_0(x_t)\,. \tag{3.17}$$

The function d belongs to C^∞ because of its relation to the smooth traces $w|_T$ and $\tilde{w}|_T$ shown above. It thus follows as in the proof of Proposition 3.2 that

$$b(\theta) \,=\, \varphi'(\theta)^2\beta_R \,-\, d'(\theta)\,, \tag{3.18}$$

and it remains to rewrite $d'(\theta)$ appropriately.

To this end, we recall from (2.11) that

$$u_0(x_t) \,=\, -\frac{1}{2\pi}\cot\frac{t-\theta}{2} \,=\, -\frac{1}{\pi}\frac{1}{t-\theta} \,+\, \frac{1}{12\pi}\,(t-\theta) \,+\, O(t-\theta)^2$$

for x_t close, but not equal to x_θ. Similarly,

$$\begin{aligned} \tilde{u}_0\big(\Phi(x_t)\big) \,&=\, -\frac{1}{\pi}\frac{1}{\varphi(t)-\varphi(\theta)} \,+\, \frac{1}{12\pi}\,(\varphi(t)-\varphi(\theta)) \,+\, O\big(\varphi(t)-\varphi(\theta)\big)^2 \\ &=\, -\frac{1}{\pi}\frac{1}{\varphi'(\theta)(t-\theta)}\left(1 - \frac{\varphi''(\theta)}{2\varphi'(\theta)}\,(t-\theta) + \frac{\varphi''(\theta)^2}{4\varphi'(\theta)^2}\,(t-\theta)^2 - \frac{\varphi'''(\theta)}{6\varphi'(\theta)}\,(t-\theta)^2\right) \\ &\quad +\, \frac{1}{12\pi}\,\varphi'(\theta)(t-\theta) \,+\, O(t-\theta)^2\,. \end{aligned}$$

It thus follows that

$$d(t) \,=\, \frac{1}{2\pi}\frac{\varphi''(\theta)}{\varphi'(\theta)} \,+\, \frac{1}{12\pi}\left(2\,\frac{\varphi'''(\theta)}{\varphi'(\theta)} \,-\, 3\,\frac{\varphi''(\theta)^2}{\varphi'(\theta)^2} \,+\, \varphi'(\theta)^2 \,-\, 1\right)(t-\theta) \,+\, O(t-\theta)^2\,,$$

which holds also for $t = \theta$ due to the smoothness of d. Hence,

$$d'(\theta) \,=\, \frac{1}{12\pi}\left(\varphi'(\theta)^2 \,-\, 1 \,+\, 2\,\frac{\varphi'''(\theta)}{\varphi'(\theta)} \,-\, 3\,\frac{\varphi''(\theta)^2}{\varphi'(\theta)^2}\right).$$

Inserting the above equation into (3.18), the assertion follows. ∎

We remark that, as expected, for a Möbius transformation of the form (3.10) the two representations (3.13) and (3.16) indeed coincide; Corollary 3.4 makes this matter more transparent.

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The backscatter data (3.16) are, in fact, the boundary values of an analytic function living in $D \setminus \overline{\Omega}$. To see this, recall that the Schwarzian derivative of an analytic function Φ, defined in a domain $G \subset \mathbb{C}$, is given by

$$S_\Phi(z) \;=\; \frac{\Phi'''(z)}{\Phi'(z)} \;-\; \frac{3}{2}\left(\frac{\Phi''(z)}{\Phi'(z)}\right)^2, \qquad z \in G\,.$$

The Schwarzian S_Φ is analytic in G if and only if Φ is locally univalent, i.e., if Φ' does not vanish anywhere in G. Moreover, S_Φ is identically zero if and only if Φ is a Möbius transformation; in a way, S_Φ measures the degree to which Φ fails to be a Möbius transformation. For more details about the Schwarzian derivative, we refer to Hille [10].

In the following corollary, x_θ is interpreted as a point in the complex plane, i.e., $x_\theta = e^{\mathrm{i}\theta}$, and Φ as an analytic function of a complex variable $z \in D \setminus \overline{\Omega}$.

COROLLARY 3.4. *Let Ω be a cavity in D, and let Φ be the conformal transformation that takes $D \setminus \overline{\Omega}$ onto $D \setminus \overline{B}_R$ for an appropriate $R \in (0,1)$. Then the associated backscatter data can (with slight abuse of notation) be extended to a function*

$$b(z) \;=\; z^2\left[\beta_R\left(\frac{\Phi'(z)}{\Phi(z)}\right)^2 + \frac{1}{6\pi}S_\Phi(z)\right] \tag{3.19}$$

holomorphic in $D \setminus \overline{\Omega}$.

Proof. A straightforward calculation on the grounds of (3.2) reveals that

$$\begin{aligned}
\Phi'(x_\theta) \;&=\; \varphi'(\theta)\, e^{\mathrm{i}(\varphi(\theta)-\theta)},\\
\Phi''(x_\theta) \;&=\; \big(-\mathrm{i}\varphi''(\theta) + \varphi'(\theta)^2 - \varphi'(\theta)\big)\, e^{\mathrm{i}(\varphi(\theta)-2\theta)},\\
\Phi'''(x_\theta) \;&=\; \big(-\varphi'''(\theta) - 3\mathrm{i}\varphi''(\theta)\varphi'(\theta) + 3\mathrm{i}\varphi''(\theta)\\
&\qquad\qquad + \varphi'(\theta)^3 - 3\varphi'(\theta)^2 + 2\varphi'(\theta)\big)\, e^{\mathrm{i}(\varphi(\theta)-3\theta)}.
\end{aligned}$$

Together with (3.2) it thus follows that

$$\left(\frac{\Phi'(x_\theta)}{\Phi(x_\theta)}\right)^2 = e^{-2\mathrm{i}\theta}\varphi'(\theta)^2$$

and

$$S_\Phi(x_\theta) = e^{-2\mathrm{i}\theta}\left(\frac{1}{2} - \frac{1}{2}\varphi'(\theta)^2 + \frac{3}{2}\,\frac{\varphi''(\theta)^2}{\varphi'(\theta)^2} - \frac{\varphi'''(\theta)}{\varphi'(\theta)}\right).$$

Substitution of these formulae in (3.16) establishes (3.19) for $z \in T$, the right-hand side being analytic in $D \setminus \overline{\Omega}$. ∎

4. The inverse problem. We now turn to the inverse problem of determining a cavity Ω from its associated backscatter data; recall that a cavity is a nonempty simply connected C^2-domain Ω with $\overline{\Omega} \subset D$, such that the conductivity σ satisfies (2.1) with $\sigma = \kappa = 0$ in Ω. We prove that such a domain is uniquely determined by its backscatter data.

THEOREM 4.1. *Assume that Ω_k, $k = 1, 2$, are two cavities in D which produce the same backscatter data $b_1 = b_2$. Then $\Omega_1 = \Omega_2$.*

Proof. Our argument is based on the results of the previous section. Accordingly, for $k = 1, 2$, let Φ_k be the conformal map which takes $D \setminus \overline{\Omega}_k$ onto the corresponding

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annulus $\{x \in D : R_k < |x| < 1\}$, and let Ψ_k be its inverse map. As mentioned above, each Φ_k (and thus Ψ_k) can be extended to a conformal mapping of some neighborhood of T, and hence,

$$\Xi = \Phi_1 \circ \Psi_2 \tag{4.1}$$

is again a well defined conformal map of some neighborhood of T, which maps T onto itself leaving its orientation invariant.

We can thus define an auxiliary holomorphic function

$$h(z) \;=\; z^2 \left[\beta_{R_1} \left(\frac{\Xi'(z)}{\Xi(z)}\right)^2 + \frac{1}{6\pi} S_\Xi(z)\right] \tag{4.2}$$

for z near T, and a comparison with (3.19) reveals (using the same abuse of notation and the same proof as in Corollary 3.4) that the boundary values of h on T are given by

$$h(\theta) \;=\; \beta_{R_1}\xi'(\theta)^2 \,+\, \frac{1}{12\pi} \,-\, \frac{1}{12\pi}\,\xi'(\theta)^2 \,+\, \frac{1}{4\pi}\,\frac{\xi''(\theta)^2}{\xi'(\theta)^2} \,-\, \frac{1}{6\pi}\,\frac{\xi'''(\theta)}{\xi'(\theta)}\,, \tag{4.3}$$

where ξ is the boundary map associated with Ξ, i.e., $\Xi(e^{\mathrm{i}\theta}) \,=\, e^{\mathrm{i}\xi(\theta)}$. It follows from the fundamental properties of the Schwarzian derivative (cf. [10]) that

$$S_\Xi = (\Psi_2')^2\,(S_{\Phi_1} \circ \Psi_2) + S_{\Psi_2}\,,$$

and, if $Id = \Phi_2 \circ \Psi_2$ denotes the identity map,

$$0 \;=\; S_{Id} \;=\; (\Psi_2')^2\,(S_{\Phi_2} \circ \Psi_2) + S_{\Psi_2}\,, \qquad \text{i.e.,} \qquad S_{\Psi_2} = -(\Psi_2')^2\,(S_{\Phi_2} \circ \Psi_2)\,.$$

Substituting these formulae in (4.2), and using (3.19), we conclude that

$$\begin{aligned} h(z) \;&=\; z^2 \Bigg[\beta_{R_1}\left(\frac{\Phi_1'\big(\Psi_2(z)\big)}{\Phi_1\big(\Psi_2(z)\big)}\Psi_2'(z)\right)^2 + \frac{1}{6\pi}\,\Psi_2'(z)^2 S_{\Phi_1}\big(\Psi_2(z)\big) \\ &\qquad\qquad - \frac{1}{6\pi}\,\Psi_2'(z)^2 S_{\Phi_2}\big(\Psi_2(z)\big)\Bigg] \\ &=\; z^2\Psi_2'(z)^2\left[\frac{b_1\big(\Psi_2(z)\big) - b_2\big(\Psi_2(z)\big)}{\Psi_2(z)^2} \,+\, \beta_{R_2}\left(\frac{\Phi_2'\big(\Psi_2(z)\big)}{\Phi_2\big(\Psi_2(z)\big)}\right)^2\right]. \end{aligned}$$

By assumption, we have $b_1 = b_2$, and hence, the above expression simplifies to

$$h(z) \;=\; z^2\Psi_2'(z)^2\,\beta_{R_2}\left(\frac{\Phi_2'\big(\Psi_2(z)\big)}{z}\right)^2 \;=\; \beta_{R_2}\,, \tag{4.4}$$

i.e., h is a constant. Inserting (4.4) into (4.3) we therefore obtain a differential equation for the derivative of the boundary map ξ, i.e.,

$$\xi'''(\theta) \;=\; \frac{3}{2}\,\frac{\xi''(\theta)^2}{\xi'(\theta)} \,+\, \gamma_2\xi'(\theta) - \gamma_1\xi'(\theta)^3\,, \tag{4.5}$$

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where we have set

$$\gamma_k \;=\; (1-12\pi\beta_{R_k})/2\,, \qquad k=1,2\,, \tag{4.6}$$

for notational convenience.

Note that we are interested only in solutions of (4.5) that are 2π-periodic and satisfy

$$\int_0^{2\pi} \xi'(\theta)\,\mathrm{d}\theta \;=\; \xi(2\pi)-\xi(0) \;=\; 2\pi,$$

for otherwise Ξ would not be a bijective transformation of T. In particular, this implies that the only admissible constant solution of (4.5) is $\xi'\equiv 1$, in which case we immediately conclude that $\gamma_1=\gamma_2$ or, equivalently, $R_1=R_2$ (cf. (2.19)). By virtue of (4.1) it follows that Φ_1 and Φ_2 agree on T up to a rotation and, using analytic continuation, thus agree up to a rotation everywhere, i.e.,

$$D\setminus\overline{\Omega}_1 \;=\; \Psi_1(D\setminus\overline{B}_{R_1}) \;=\; \Psi_2(D\setminus\overline{B}_{R_2}) \;=\; D\setminus\overline{\Omega}_2\,,$$

which was to be shown.

In the remainder of this proof we can therefore assume that ξ' is not a constant; we denote by θ^* and θ_* some locations of the global maximum, respectively, minimum of ξ' over $[0,2\pi]$. As pointed out in section 3, ξ' can be extended to an analytic and 2π-periodic function over all of $\mathbb{R}$, which means, in particular, that

$$\xi''(\theta^*) \;=\; \xi''(\theta_*) \;=\; 0 \qquad \text{and} \qquad \xi'''(\theta^*) \;\le\; 0 \;\le\; \xi'''(\theta_*)\,. \tag{4.7}$$

Moreover, as Ξ leaves the orientation of T invariant, ξ' is a strictly positive function. We therefore conclude from (4.5) and (4.7) that

$$\gamma_2\bigl(\xi'(\theta_*)^{-2}-\xi'(\theta^*)^{-2}\bigr) \;=\; \frac{\xi'''(\theta_*)}{\xi'(\theta_*)^3} \;-\; \frac{\xi'''(\theta^*)}{\xi'(\theta^*)^3} \;\ge\; 0\,, \tag{4.8}$$

from which we deduce that γ_2 is a nonnegative number. A similar argument can be used to reveal that $\gamma_1\ge 0$ as well, but we will do somewhat better below.

Without loss of generality we may assume that $\theta^*=0$, which provides two initial conditions

$$\xi'(0) \;=\; c\,, \qquad \xi''(0) \;=\; 0 \tag{4.9}$$

for ξ'. Since c is nonzero, the initial value problem (4.5), (4.9) has a unique (local) solution ξ', depending only on the free parameter $c>0$, and the two numbers γ_1 and γ_2 from (4.6), of course. A somewhat tedious, but straightforward, computation reveals that this solution has the explicit form

$$\xi'(\theta) \;=\; \frac{c\gamma_2}{c^2\gamma_1\sin^2(\omega\,\theta)\;+\;\gamma_2\cos^2(\omega\,\theta)}\,, \qquad \omega=\sqrt{\gamma_2/2}\,, \tag{4.10}$$

provided that $\gamma_2>0$, and $\xi'\equiv c$ if $\gamma_2=0$, respectively. To see the latter, we note that for $\gamma_2=0$ we must have $\xi'''(0)=0$ in (4.8), and hence, $\gamma_1=0$ by virtue of (4.5). As is easy to see, $\xi'\equiv c$ is then the unique solution of (4.5), (4.9).

As the constant case has been excluded above, we can restrict our attention to the case in which $\gamma_2>0$ and ξ' is given by (4.10). We first note that we must then

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also have $\gamma_1 > 0$, for ξ' would encounter singularities otherwise. Moreover, as ξ' is 2π-periodic, we conclude from (4.10) that ω is a half integer, but

$$0 \,<\, 2\omega \,=\, \sqrt{2\gamma_2} \,=\, (1 - 12\pi\beta_{R_2})^{1/2} \,<\, 1$$

by virtue of (4.6) and (2.19); this is a contradiction. It thus follows that the nonconstant solution (4.10) cannot occur, and the proof is complete. ∎

A straightforward modification of this argument extends the result to the case when one of the two domains is the empty set.

COROLLARY 4.2. *Every cavity $\Omega \subset D$ has nontrivial backscatter data $b \not\equiv 0$.*

Proof. Assume that Ω produces vanishing backscatter data $b \equiv 0$. Let Φ_2 be the conformal map which takes $D \setminus \overline{\Omega}$ onto the corresponding annulus, and let Ψ_2 be its inverse map. We set $\Phi_1 = Id$ and $\beta_{R_1} = 0$ so that (3.19) holds with left-hand side $b_1 \equiv 0$. Then, if we define Ξ as in (4.1), we can follow the proof of Theorem 4.1 to obtain the desired contradiction. ∎

Our results readily extend to the case of "limited angle" data, as made precise in the following theorem.

THEOREM 4.3. *Theorem* 4.1 *and Corollary* 4.2 *remain true if the backscatter data are only given on some nonempty open subinterval $\mathcal{I} \subset [0, 2\pi)$.*

Proof. According to Corollary 3.4 the backscatter data are the trace on T of a holomorphic function defined in some neighborhood of T. As such, this holomorphic function is uniquely defined by its values on a nondegenerate arc on the unit circle. Thus, if the backscatter data of two cavities (or one cavity and the homogeneous body) agree on such an arc, then they agree on the whole unit circle, and thus Theorem 4.1 and Corollary 4.2 apply. ∎

Finally, we want to point out that the backscatter problem remains uniquely solvable if stated in a more general domain. To this end, let $\tilde{D} \subset \mathbb{R}^2$ be a smooth and simply connected bounded domain and $\tilde{\Omega}$ a cavity inside $\tilde{D}$ with C^2-boundary $\tilde{\Gamma} = \partial\tilde{\Omega} \subset \tilde{D}$. The concept of backscatter data introduced in section 2 for the unit disk has a natural counterpart on $\tilde{T} = \partial\tilde{D}$. Let $\tilde{u}$ be the solution of

$$\begin{aligned} \Delta\tilde{u} \,&=\, 0 \quad \text{in } \tilde{D} \setminus \overline{\tilde{\Omega}}, \qquad & \frac{\partial}{\partial\nu}\,\tilde{u} \,=\, 0 \quad \text{on } \tilde{\Gamma}, \\ \frac{\partial}{\partial\nu}\,\tilde{u} \,&=\, \frac{\partial}{\partial\tau}\,\delta(\,\cdot\, - y) \quad \text{on } \tilde{T}, \qquad & \int_{\tilde{T}} \tilde{u}\,\mathrm{d}s \,=\, 0\,, \end{aligned} \tag{4.11}$$

and let $\tilde{u}_0$ be the corresponding reference potential. Then, in the spirit of (2.15) we define

$$\tilde{b}(y) \,=\, -\left(\frac{\partial}{\partial\tau}\,\tilde{w}|_{\tilde{T}}\right)(y), \qquad y \in \tilde{T},$$

where

$$\tilde{w} \,=\, \tilde{u} - \tilde{u}_0\,.$$

Note that the tangential derivative of the delta distribution in (4.11) is defined by replacing T by $\tilde{T}$ and x_θ by y in (2.3).

THEOREM 4.4. *Let $\tilde{\Omega}_k$, $k = 1, 2$, be two (possibly empty) cavities in the simply connected domain $\tilde{D} \subset \mathbb{R}^2$ that has a C^∞-boundary. Assume that the corresponding backscatter data $\tilde{b}_k$, $k = 1, 2$, coincide on a nonempty, relatively open subset of $\tilde{T}$. Then $\tilde{\Omega}_1 = \tilde{\Omega}_2$.*

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Proof. According to the Riemann mapping theorem, there exists a conformal map Υ that takes the unit disk D onto $\tilde{D}$. Moreover, Υ has a C^∞-extension to the closure of D (cf., e.g., [19, section 3.3]). We define $\Omega_k = \Upsilon^{-1}(\tilde{\Omega}_k)$, $k = 1, 2$, and denote the backscatter data corresponding to D and Ω_k by b_k. Now, the same line of reasoning that led to the formula (3.13) of Proposition 3.2 gives the representations

$$b_k(\theta) \,=\, |\Upsilon'(x_\theta)|^2\, \tilde{b}_k(\Upsilon(x_\theta)), \qquad k = 1, 2\,.$$

In particular, b_1 and b_2 coincide on some nondegenerate interval of $[0, 2\pi)$, which implies, according to Theorem 4.3, that $\Omega_1 = \Omega_2$. Since

$$\tilde{\Omega}_1 = \Upsilon(\Omega_1) = \Upsilon(\Omega_2) = \tilde{\Omega}_2,$$

the proof is complete. □

5. Concluding remarks. In the uniqueness considerations of section 4 we restricted our attention to insulating cavities. However, the same results hold also for ideally conducting inclusions—even the proofs are essentially the same. On the other hand, there seems to be no straightforward way of generalizing our arguments to the case of inhomogeneities with known (positive and finite) constant conductivities.

Appendix. In the following, we show the connection between the backscatter data and measurements taken with two small electrodes placed close to each other. To simplify the argument, we restrict ourselves to the unit disk D (as in section 2) and choose $\theta = 0$, so that $x_\theta = (1, 0)$. Let two electrodes of length $h > 0$ be attached to the boundary of D at $\gamma_+ = \gamma_+^h = \{x_t : t \in [h/2, 3h/2]\}$ and $\gamma_- = \gamma_-^h = \{x_t : x_{-t} \in \gamma_+\}$, which are used to drive an electric current from γ_- to γ_+. In the so-called gap model (cf. [11]), which is a very simple but not totally accurate electrode model, the corresponding potential u_h is taken to satisfy (2.6) with f_n replaced by

$$f_h \,:=\, -\frac{1}{2h^2}\left(\chi_{\gamma_+} - \chi_{\gamma_-}\right) \,\in\, \mathcal{L}^2_\diamond(T)\,,$$

where χ_A denotes the characteristic function of the set A. Note that the current has been normalized appropriately so that the limit as $h \to 0$ is nontrivial; see (A.5).

Denoting by $u_{h,0}$ the corresponding reference potential, i.e.,

$$u_{h,0}(z) \,=\, \int_T N(z, x_t)\, f_h(x_t)\; \mathrm{d}s(x_t)\,, \qquad z \in D\,, \tag{A.1}$$

we write the relative potential $w_h = u_h - u_{h,0}$ as a single layer potential over Γ (cf. (2.12)) with density $\psi_h \in C_\diamond(\Gamma)$:

$$w_h(z) \,=\, \int_\Gamma N(z, x)\psi_h(x)\; \mathrm{d}s(x)\,, \qquad z \in D \setminus \Gamma\,. \tag{A.2}$$

Here, ψ_h is the solution of (2.13) with u_0 replaced by $u_{h,0}$. According to the gap model, the relative potentials measured at the two electrodes $\gamma_\pm$ equal

$$W_\pm \,:=\, \frac{1}{h}\int_{\gamma_\pm} w_h(x_t)\; \mathrm{d}s(x_t)\,,$$

respectively. We are going to verify the relation

$$W_- - W_+ \,=\, -2h\,\left(\frac{\partial}{\partial\tau} w|_T\right)(x_0) + O(h^3)\,, \tag{A.3}$$

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where w is given by (2.12) and (2.13). Throughout the following argumentation, C denotes a generic constant depending only on Γ.

To begin with, we show that the family of boundary currents f_h converges toward $f := \frac{\partial}{\partial \tau}\delta(\cdot - x_0)$ in $H^{-4}(T)$ as h tends to zero. The Sobolev embedding theorem (cf., e.g., Saranen and Vainikko [21, Lemma 5.3.3]) assures that, for any $v \in H^4(T)$, the angular derivative $\partial v/\partial s \in H^3(T)$ is twice continuously differentiable and satisfies

$$\sum_{l=0}^{3} \sup_{\mu\in[0,2\pi)} \left| \frac{\partial^l v}{\partial s^l}(x_\mu) \right| \le C \, \|v\|_{H^4(T)} \, .$$

Therefore, it is admissible to approximate $\partial v/\partial s$ by central differences (cf. [6, Lemma 4.1.1]), and we arrive at

$$\sup_{|t|\le 3h/2} \left| \frac{v(x_t) - v(x_{-t})}{2t} - \frac{\partial v}{\partial s}(x_0) \right| \le Ch^2 \, \|v\|_{H^4(T)}.$$

The evaluation of f_h at $v \in H^4(T)$ can, therefore, be estimated as

$$\begin{aligned} \left| \langle f_h, v \rangle + \frac{\partial v}{\partial s}(x_0) \right| &\le \frac{1}{2h^2} \int_{h/2}^{3h/2} 2t \left| \frac{v(x_t) - v(x_{-t})}{2t} - \frac{\partial v}{\partial s}(x_0) \right| \mathrm{d}t \\ &\le Ch^2 \|v\|_{H^4(T)} \, . \end{aligned} \tag{A.4}$$

Hence, it follows from an obvious analogue of (2.7) and (A.4) that

$$\|f_h - f\|_{H^{-4}(T)} \le Ch^2 \, . \tag{A.5}$$

Next, we are going to show that the densities ψ_h converge to ψ of (2.13) in the topology of $C(\Gamma)$. We deduce from (A.1) that

$$\frac{\partial u_{h,0}}{\partial \nu}(z) = \left\langle f_h, \frac{\partial}{\partial_z \nu} N(z, \cdot\,) \right\rangle$$

for $z \in \Gamma$. As $\partial N(z, \cdot\,)/\partial_z \nu$ is uniformly bounded in $H^4(T)$ over $z \in \Gamma$, we obtain from (A.4) by replacing v with $\partial N(z, \cdot\,)/\partial_z \nu$ and in view of (2.10) that

$$\left\| \frac{\partial u_{h,0}}{\partial \nu} - \frac{\partial u_0}{\partial \nu} \right\|_{C(\Gamma)} \le Ch^2 \, .$$

Furthermore, because the operator on the left-hand side of (2.13) is invertible as a map from $C_\diamond(\Gamma)$ onto itself by virtue of the Fredholm alternative, it follows that

$$\|\psi_h - \psi\|_{C(\Gamma)} \le Ch^2 \, ,$$

which is the needed estimate.

Using the representations (2.12) and (A.2), we deduce that

$$\left\| \frac{\partial w_h}{\partial \tau} - \frac{\partial w}{\partial \tau} \right\|_{C(T)} \le \sup_{z\in T} \int_\Gamma \left| \frac{\partial}{\partial_z \tau} N(z,x) \right| \mathrm{d}s(x) \; \|\psi_h - \psi\|_{C(\Gamma)} \le Ch^2 \, . \tag{A.6}$$

Moreover, by differentiating under the integral sign in (A.2), we see that all tangential derivatives of $w_h|_T$ are well defined and, in particular, $w_h|_T$ remains uniformly

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bounded in $H^4(T)$ as $h \to 0$. As a consequence, it follows that

$$\frac{W_- - W_+}{2h} = -\frac{1}{2h^2} \int_{\gamma_+} \bigl(w_h(x_t) - w_h(x_{-t})\bigr)\, \mathrm{d}s(x_t) = \langle\, f_h, w_h \,\rangle$$
$$= \langle\, f_h - f, w_h \,\rangle - \left(\frac{\partial}{\partial \tau} w_h|_T\right)(x_0)\,,$$

and thus

$$\left|\frac{W_- - W_+}{2h} + \left(\frac{\partial}{\partial \tau} w|_T\right)(x_0)\right| \le C\,\|f_h - f\|_{H^{-4}(T)} + \left\|\frac{\partial w_h}{\partial \tau} - \frac{\partial w}{\partial \tau}\right\|_{C(T)},$$

which, together with (A.5) and (A.6), finally establishes (A.3).

In the framework of the gap model, $2hf_h$ corresponds to one unit of current flowing from γ_- to γ_+. In view of (2.15) we have thus justified the following interpretation of the backscatter data for this model: For a unit current flowing from γ_- to γ_+, the voltage difference between the electrodes γ_- and γ_+ equals $4h^2 b(0) + O(h^4)$, where the constant hidden in the $O(\cdot)$-notation depends on Γ and thus on Ω. Therefore, the backscatter data can be recovered from the leading term of the asymptotic expansion of the measured data in the framework of the gap model.

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