This is the program for the workgroup on "Néron models of curves and abelian varieties" at the Johannes-Gutenberg Universitat of Mainz. Below you will find an outline of the topics treated in the talks and the bibliography for this work-group.

1 Introduction

We introduce the main purpose of this seminar, and explain why studying Néron models is interesting and useful. For instance, we mention Faltings' proof of (some of) the Tate conjectures for abelian varieties and Wiles' proof of Fermat's last theorem. Then we explain what we mean by a *model* of a variety over K, where K is the fraction field of a Dedekind scheme B. Then we will work our way towards a definition of the Néron model. That constitutes the first part of this talk. We will explain Néron's smoothening process (Talk 4.1), the weak Néron mapping property and weak Néron models (Talk **??**).

In the second part we discuss how to prove that abelian schemes over discrete valuation rings are Néron models of their generic fibre. To do this, we state Weil's extension theorem (see Talk 4.3). In particular, elliptic curves over $\mathbb{Z}_p$ and $\mathbb{C}[[t]]$ are Néron models of their generic fibre. Also, using Weil's extension theorem again, we prove that smooth proper models of abelian varieties have the Néron mapping property (assuming the existence of a Néron model).

We will give an overview of the construction of a Néron model of an abelian variety.

Speaker: Ariyan Javanpeykar

2 Elliptic curves

We start with an introduction to the arithmetic of elliptic curves. Let E be an elliptic curve over a field K. We define this to be a pair (E, o) with E a smooth proper geometrically connected curve of genus one over K and $o \in E(K)$. We prove that E can be given by a Weierstrass equation. Then we focus on Weierstrass models, and the discriminant of a Weierstrass model. Several examples are presented, and many pictures are drawn. Finally, we introduce Weierstrass models over Dedekind schemes and study the valuation of the associated discriminant (which is easily shown to be an invariant of the Weierstrass model). We state that an elliptic curve E/K has a smooth proper model over R if and only if its minimal discriminant over R is a unit in R. Finally, we work out an explicit example of an elliptic curve with multiplicative reduction of type I_n. Namely, let π be a uniformizer for R and E be given by $y^2 = x^3 + x^2 + \pi^n$ over R.

Next we focus on Néron models of elliptic curves (without having fully proven their existence actually). In particular, we discuss the Kodaira-Néron classification of the fibres of the minimal regular model. The aim of this talk is to present many examples.

We present Tate's algorithm for computing the special fibre of the Néron model.

We will do a special session on SAGE to allow everyone to do as many examples as they please. The exercises for the SAGE session can be found on the website.

Speaker: Sonia Samol, Weierstrass models and discriminants.

Speaker: Ariyan Javanpeykar, Kodaira-Néron classification, Tate's algorithm and SAGE session.

3 Models of curves

We start our proof of the existence of the minimal regular model of a curve of positive genus over a discretely valued field.

Theorem 3.1. *Let B be a Dedekind scheme with function field K and X a curve of positive genus over K. Then X has a unique minimal regular model over B.*

In the first part we prove Lipman's theorem on desingularization of (excellent integral normal) two-dimensional schemes and deduce that X has a regular proper model over B.

We introduce the notion of minimality and prove that curves of positive genus have minimal regular models; see Section 9 in [6]. To achieve this we need Lipman's desingularisation theorem ([4] and [5]) and Castelnuovo's contraction theorem ([6]).

As an application, we prove that the smooth locus of the minimal regular model of an elliptic curve is a Néron model.

If time permits, as an example, we construct the minimal regular model of the Fermat curve $x^p + y^p = z^p$ over $\mathbf{Z}[\zeta_p]$ for $p \geq 3$ a prime.

Speaker: Ronan Terpereau on Lipman's desingularization theorem

Speaker: Axel Stäbler on minimal regular models

4 Néron models of abelian varieties

4.1 Smoothening

Let X be a smooth proper variety over a discretely valued field K. Let O_K be the ring of integers of K and let $\mathcal{X} \to \operatorname{Spec} O_K$ be a flat proper model for A. In this talk, we explain the smoothening process. We then prove that X has a smooth model satisfying the Néron mapping property for étale algebras over O_K. Moreover, we prove that any smooth model of a smooth proper variety satisfying the Néron mapping property for étale algebras over O_K is a "weak Néron model". Thus, smooth proper varieties over K have weak Néron models over B. References are [1] and [2], see also [3].

Speaker: Thomas Weißchuh

4.2 Non-minimal components

We show that every abelian variety A over K has a smooth model with a "birational group law". This is done by removing non-minimal components from a weak Néron model.

References are [2] and [1, Chapter 4].

Speaker: Abolfazl Mohajer

4.3 Weil's theorem

We explain how a model $\mathcal{A}_3$ over O_K for A with a birational group law can be extended to a group scheme over O_K (after a base-change to a strict henselisation of O_K).

Then we use descent theory to obtain a group scheme over O_K extending $\mathcal{A}_3$. By Weil's extension theorem we conclude that this group scheme is the Néron model of A.

We follow Artin-Edixhoven-Romagny; see [2], but also [1, Chapter 5 and Chapter 6].

Speaker: Bas Edixhoven

5 Néron models of non-proper group schemes

We discuss Néron models of non-necessarily proper group schemes over a Dedekind scheme B. Such Néron models aren't necessarily of finite type in general, but only locally of finite type. Therefore, we work instead with lft-Néron models.

We show that tori have Néron models, and that unipotent groups (assuming B has perfect residue fields) do not have Néron models. Interestingly, the "globalizing" of a Néron model in char $p > 0$ is not a straightforward matter, and we present Oesterle's example to illustrate this.

We follow [1, Chapter 10].

Speaker: Ronan Terpereau

6 The Picard functor and the relative Jacobian

As a first application of the theory of Néron models we study the Picard functor and the relative Jacobian; see [1, Chapter 8 and Chapter 9]. (Other useful references are the articles of Cedric Pepin [10] and [11].)

Speaker: Abolfazl Mohajer

7 Liu-Tong theorem on Néron models

We prove a recent result of Qing Liu and Jilong Tong that all curves of positive genus over a discretely valued field admit a Neron model and that the smooth locus of the minimal regular model is the Neron model. References are [6], [7], [8], and the preprint of Liu-Tong [9].

Speaker:

8 Non-existence of Néron models

We will discuss the non-existence of Néron models for abelian varieties over higher-dimensional base schemes. This talk will be given by David Holmes (Leiden University).

References

[1] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.

[2] B. Edixhoven and M. Romagny. Group schemes out of birational group laws, Néron models. *On Romagny's website.*

[3] Hélène Esnault and Johannes Nicaise. Finite group actions, rational fixed points and weak Néron models. *Pure Appl. Math. Q.*, 7(4, Special Issue: In memory of Eckart Viehweg):1209–1240, 2011.

[4] Joseph Lipman. Rational singularities, with applications to algebraic surfaces and unique factorization. *Inst. Hautes Études Sci. Publ. Math.*, (36):195–279, 1969.

[5] Joseph Lipman. Desingularization of two-dimensional schemes. *Ann. Math. (2)*, 107(1):151–207, 1978.

[6] Q. Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[7] Q. Liu. Stable reduction of finite covers of curves. *Compos. Math.*, 142(1):101–118, 2006.

[8] Q. Liu and D. Lorenzini. Models of curves and finite covers. *Compositio Math.*, 118(1):61–102, 1999.

[9] Q. Liu and J. Tong. Néron models of curves. *arXiv:1312.4822*.

[10] Cédric Pépin. Néron's pairing and relative algebraic equivalence. *Algebra Number Theory*, 6(7):1315–1348, 2012.

[11] Cédric Pépin. Modèles semi-factoriels et modèles de Néron. *Math. Ann.*, 355(1):147–185, 2013.

[12] Jean-Pierre Serre and John Tate. Good reduction of abelian varieties. *Ann. of Math. (2)*, 88:492–517, 1968.