Seminar: Néron models

We study Néron models of curves and abelian varieties. The main goal is to familiarize students with some basic notions of algebraic geometry. We will do this by (i) proving that every abelian variety admits a Néron model, and (ii) using the Néron model in several different contexts. Before studying models of abelian varieties in this workgroup, we treat models of curves.

Let X be a smooth projective curve of positive genus over a discretely valued field K. Let O_K be the ring of integers of K. A model of X over O_K is a flat projective scheme $\mathcal{X}$ whose generic fiber is isomorphic to X over K. Note that X always has a model over O_K. One can ask whether X has a "canonical" model over O_K satisfying some "good" properties. The answer to this question is provided by the existence of the minimal regular model. This is a regular projective model of X over O_K satisfying some "minimality" condition. In this seminar we will start by proving the existence of the minimal regular model of X. To make things explicit, we will treat the case of elliptic curves in more detail.

At the end of this seminar, we will see that the smooth locus of the minimal regular model of an elliptic curve has a very interesting property. Namely, it is a "Néron model". Recently, Qing Liu and Jilong Tong have generalized this theorem: the smooth locus of the minimal regular model of a curve of positive genus is a Néron model.

The importance of studying "models" of curves can be found, for example, in the classical work of Arakelov, Manin, Parshin and Szpiro on the Mordell conjecture. Also, it is used in Faltings' famous proof of the Tate, Shafarevich and Mordell conjectures We will not treat these topics in this seminar.

In the second part of this workgroup we will spend a considerable amount of time on studying models of abelian varieties with nice properties. This leads us to the notion of Néron models. Let A be an abelian variety over K. If A is an elliptic curve, the theory of minimal regular models of curves implies that A has a "Néron model". It is a result of André Néron that A always has a "Néron model". This means that there exists a smooth scheme $\mathcal{A}$ over O_K whose generic fiber is isomorphic to A over K and which satisfies the Néron mapping property : for all smooth schemes $\mathcal{Z}$ over O_K, the canonical morphism of sets

$$\mathrm{Hom}(\mathcal{Z}, \mathcal{A}) \to \mathrm{Hom}(\mathcal{Z}_K, A)$$

is bijective. Note that the Néron model $\mathcal{A}$ is in fact a smooth commutative group scheme over O_K, and that it is unique up to a unique isomorphism. Let us briefly explain how one can construct the Néron model in five steps.

1. Let $\mathcal{A}_0$ be any flat projective model of A (over O_K). This exists by the fact that A is projective over K: take the Zariski closure of $A \subset \mathbf{P}^n_K$ in $\mathbf{P}^n_{O_K}$, where $A \subset \mathbf{P}^n_K$ is some closed immersion of A into some projective space.

2. Let $\mathcal{A}_1$ be a "smoothening" of $\mathcal{A}_0$. This is a model for A that comes with a proper morphism $\mathcal{A}_1 \to \mathcal{A}_0$ and the property that every $(O_K)^{sh}$-point of $\mathcal{A}_0$ lifts to a point

in the smooth locus of $\mathcal{A}_1$, where $(O_K)^{sh}$ denotes the strict henselisation of O_K (with respect to some choice of separable closure of the residue field of O_K).

3. Let $\mathcal{A}_2$ be the smooth locus of $\mathcal{A}_1$ (with respect to the morphism $\mathcal{A}_1 \to \operatorname{Spec} O_K$). Since $\mathcal{A}_2$ satisfies the Néron mapping property for étale algebras over O_K, it satisfies the "Weak Néron mapping property": for all smooth schemes $\mathcal{Z}$ over O_K, any K-rational map $\mathcal{Z}_K - \to A$ extends uniquely to an O_K-rational map $\mathcal{Z} - \to \mathcal{A}_2$.

4. To obtain the Néron model, let $\mathcal{A}_3$ be the complement of all "non-minimal components" of $\mathcal{A}_2$. Then, $\mathcal{A}_3$ is not quite a group scheme in general, but it comes with a "birational group law".

5. By a theorem of Weil (Weil's extension theorem for birational group laws) there exists a separated smooth group scheme $\mathcal{A}_4$ of finite type containing $\mathcal{A}_3$ as an O_K-dense open subscheme and whose group law extends the "birational group law" on $\mathcal{A}_3$. The construction of $\mathcal{A}_4$ is quite difficult: its existence is first proven after base-change to $(O_K)^{sh}$ and then descent theory is applied to construct $\mathcal{A}_4$ over O_K. Finally, by applying Weil's purity theorem for morphisms to group schemes, we conclude that $\mathcal{A}_4$ is the Néron model of A.

The construction of the Néron model of an abelian variety (and a curve of positive genus) requires a firm understanding of several different notions in algebraic geometry : smoothness, étale algebras, resolution of singularities, descent theory, smoothening processes, purity theorems, birational group laws and group schemes. In this seminar we will treat these notions extensively.

The importance of Néron models arises in the proofs (due to Faltings) of the isogeny, Shafarevich, and Tate conjectures for abelian varieties. Moreover, Néron models appear in the study of motivic Serre invariants (Esnault, Halle, Nicaise, et al.). In this workgroup we will not touch upon these subjects as they would lead us too far astray from our main topic. Fortunately, there are other applications which are easier to grasp. In fact, we will finish with two beautiful applications of the theory of Néron models for abelian varieties.

1. We will prove the Néron-Ogg-Shafarevich criterion for good reduction.

2. We will treat the relation between the Picard functor and the relative Jacobian of a curve.

3. We will prove that all curves of positive genus have Néron models (Liu-Tong, 2013).