HW 7: Algebraische Geometrie II

Handing in: Hand in by February 8th 2016. Grades are from 1 to 10. Keep a copy of your work and include an email address + home address to which we can return the graded exercises. You are strongly encouraged to write your solutions in TeX and hand in a printout. If you write your solutions by hand, write legibly! Parts that we cannot easily decipher will simply be ignored.

The rules of the game: Your solutions to these exercises should convince us that you have a good understanding of what has been discussed so far. This involves more than just getting the correct answers, which, anyway, might be obtained from various internet sources or the brain of a fellow student. You may use everything that has been discussed so far in the course. Give full details for your arguments, with precise references for the results you use. What you hand in should be your own, individual work.

Exercise 1 (2 points)**.** Let X be a smooth projective irreducible curve of genus zero over an algebraically closed field k. In this exercise we show that X is isomorphic to $\mathbb{P}^1(k)$. Let P, Q, and R be distinct points in X.

1. Using Riemann-Roch and Serre duality, show that there is a unique f in $K(X)^\times$ such that $\operatorname{div}(f) = P - R$ and $f(Q) = 1$.
2. Show that the morphism of k-algebras $k[x] \to \mathcal{O}_X(X - \{R\})$ that sends x to f is an isomorphism, and that the morphism $k[x^{-1}] \to \mathcal{O}_X(X - \{P\})$ that sends x^{-1} to f^{-1} is an isomorphism.
3. Conclude that the rational function f induces an isomorphism $f : X \to \mathbb{P}^1(k)$.

Exercise 2 (2 points)**.** Write down an elliptic curve E over $\mathbb{F}_2$ with $E(\mathbb{F}_2)$ the trivial group.

Exercise 3 (2 points)**.** Prove or disprove:

1. Let $\sigma : X \to X$ be a non-trivial automorphism of a smooth projective irreducible curve of genus at least two over an algebraically closed field k. Let $\Gamma_\sigma \subset X \times X$ be the graph of σ. Then $\Delta \cdot \Gamma_\sigma \neq \Delta \cdot \Delta$.
2. If E is an elliptic curve over $\mathbb{F}_5$, then $E(\mathbb{F}_5)$ is a non-trivial group.

Exercise 4 (2 points)**.** Let k be an algebraically closed field. Let X be a smooth projective irreducible curve of genus 2 over k with canonical divisor K_X. For D a divisor on X, prove that

$$\dim_k \mathrm{H}^0(X, D) = \begin{cases} \deg D - 1 & \deg D \geq 3 \\ 2 & D \equiv K_X \\ 1 & \deg D = 2, D \not\equiv K_X \\ 0 & \deg D < 0 \end{cases}.$$

(Here $D \equiv K_X$ means that $D - K_X$ is the divisor of some non-zero rational function on X.)

Exercise 5 (2 points)**.** Let k be an algebraically closed field with $6 \in k^\times$ and let C be the smooth projective plane curve having affine Weierstrass equation $y^2 = f(x)$, where f is a polynomial of degree 3 in $k[x]$ with distinct roots. Consider the rational 1-form $\omega = \frac{dx}{2y}$ on C. Show that $\mathrm{div}(\omega) = 0$ and deduce that $g(C) = 1$.