HW 5: Algebraische Geometrie II

Handing in: Hand in by January 18th 2016. Grades are from 1 to 10. Keep a copy of your work and include an email address + home address to which we can return the graded exercises. You are strongly encouraged to write your solutions in TeX and hand in a printout. If you write your solutions by hand, write legibly! Parts that we cannot easily decipher will simply be ignored.

The rules of the game: Your solutions to these exercises should convince us that you have a good understanding of what has been discussed so far. This involves more than just getting the correct answers, which, anyway, might be obtained from various internet sources or the brain of a fellow student. You may use everything that has been discussed so far in the course. Give full details for your arguments, with precise references for the results you use. What you hand in should be your own, individual work.

Let k be an algebraically closed field.

Exercise 1. Let X be a smooth projective irreducible curve over k. Let $f : X \to \mathbb{P}^1(k)$ be a morphism of varieties.

1. Show that f is either constant or surjective.
2. Let U be the complement of $f^{-1}\{(1:0)\}$. Assume that U is non-empty. Show that $f|_U$, seen as a map from U to $\mathbb{A}^1(k)$, defines an element $\widetilde{f}$ of the field $K(X)$ of rational functions of X.
3. Show that $f \mapsto \widetilde{f}$ defines a bijection between the set of morphisms $X \to \mathbb{P}^1(k)$ whose image is not $\{(1:0)\}$ and $K(X)$.
4. Let $X = \mathbb{P}^1(k)$ and $f : X \to \mathbb{P}^1(k)$ an isomorphism of varieties. Show that there exist $a, b, c, d \in k$ such that
$$\widetilde{f} = \frac{ax+b}{cx+d},$$
where we have identified $K(\mathbb{P}^1(k))$ with the field of fractions of $k[x] = \mathcal{O}_{\mathbb{P}^1(k)}(\mathbb{A}^1(k))$. Deduce that $\mathrm{PGL}_2(k)$ is the group of automorphisms of $\mathbb{P}^1(k)$.

Exercise 2. Consider the rational 1-form $x^{-1}dx$ on $\mathbb{P}^1(k)$. Compute its order and residue at all P in $\mathbb{P}^1(k)$. (Here we identify $K(\mathbb{P}^1(k))$ with the fraction field of $k[x]$.)

Exercise 3. Let X be the affine curve over k defined by the equation $y^2 = x^3 + 2$ over k. Compute the dimension of the tangent space at each P in X.