HW 4: Algebraische Geometrie II

Handing in: Hand in by December 14th 2015. Grades are from 1 to 10. Keep a copy of your work and include an email address + home address to which we can return the graded exercises. You are strongly encouraged to write your solutions in TeX and hand in a printout. If you write your solutions by hand, write legibly! Parts that we cannot easily decipher will simply be ignored.

The rules of the game: Your solutions to these exercises should convince us that you have a good understanding of what has been discussed so far. This involves more than just getting the correct answers, which, anyway, might be obtained from various internet sources or the brain of a fellow student. You may use everything that has been discussed so far in the course. Give full details for your arguments, with precise references for the results you use. What you hand in should be your own, individual work.

Let k be an algebraically closed field.

Exercise 1. Let X be a smooth projective connected curve over k and $P \in X(k)$. Use Riemann-Roch to show that the k-vector space $\mathcal{O}_X(U)$ of regular functions on $U := X - \{P\}$ is infinite-dimensional.

Exercise 2. We consider divisors on $\mathbb{P}^1(k)$.

1. Show that, for every $P \in \mathbb{P}^1(k)$, there exists a rational function f on $\mathbb{P}^1(k)$ such that the divisor $\mathrm{div}(f)$ of f satisfies $\mathrm{div}(f) = P - \infty$.
2. Compute a k-basis for $\mathrm{H}^0(\mathbb{P}^1(k), n \cdot [0])$, where $n \geq 1$ is an integer.

Exercise 3. Let $n \geq 0$ be an integer. Show that $\mathbb{P}^n(k)$ is affine if and only if $n = 0$.

Exercise 4. Let q and n be positive integers.

1. Show that the morphism of sets

$$f : \mathbb{P}^n(k) \to \mathbb{P}^n(k), \quad (a_0 : \ldots : a_n) \mapsto (a_0^q : \ldots : a_n^q)$$

   is a morphism of varieties over k.

2. Assume now that k has characteristic $p > 0$ and that $q = p^d$ for some integer $d > 0$. Show that f is a bijection of sets, but not an isomorphism of varieties.
3. Still assuming k has characteristic $p > 0$ and that $q = p^d$ for some integer $d > 0$, find all $P \in \mathbb{P}^n(k)$ such that $f(P) = P$.

Exercise 5. Prove or disprove (by means of a counterexample): If k is an algebraically closed field and n is an integer, then the affine curve X defined by the equation $y^2 = x^3 + n$ is smooth if and only if n is invertible in k.

Exercise 6. Let

$$\psi : \mathbb{P}^{m-1}(k) \times \mathbb{P}^{n-1}(k) \to \mathbb{P}^{nm-1}(k)$$

be the map of sets

$$\psi((a_1 : \ldots : a_m), (b_1 : \ldots : b_n)) = (a_1 b_1 : \ldots : a_m b_n).$$

Let $X \subset \mathbb{P}^{m-1}(k)$ and $Y \subset \mathbb{P}^{n-1}(k)$ be closed.

1. Show that the image of ψ is closed. (We endow henceforth the image of ψ with its sheaf of regular functions.)
2. Show that ψ induces an isomorphism of varieties from $\mathbb{P}^{m-1}(k) \times \mathbb{P}^{n-1}(k)$ to its image.
3. Show that ψ restricts to an isomorphism from the product variety $X \times Y$ to the projective variety $\psi(X \times Y)$.
4. Show that the diagonal $\Delta_{\mathbb{P}^{n-1}(k)}$ is closed in $\mathbb{P}^{n-1}(k) \times \mathbb{P}^{n-1}(k)$.
5. Show that projective varieties are separated.